Rewritable Groups

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Polynomial Identities

We say that group G satisfies PI_n if its group algebra $K[G]$ satisfies a polynomial identity of degree n. Of course, this depends somewhat on the field K.

Kaplansky (1949) observed that if G has an abelian subgroup A of finite index n, then $K[G]$ satisfies the standard identity s_{2n} and hence G satisfies PI_{2n}. We seek a converse of the form: If G satisfies PI_n, then G has an abelian subgroup A of index $\leq f(n)$.

Assume K has characteristic 0. If $n \leq 5$, Amitsur (1961) proved such a result using central polynomials. Only Wagner's polynomial (1937) for 2×2 matrices was known at that time. Then Isaacs and I (1964) proved the general result using the character theory of finite groups.

Characteristic $p > 0$

Now let K have characteristic $p > 0$. M. Smith (1971) in her thesis, used certain "linear identities" to obtain strong partial results on the converse. Building on this, and using more group theory, I obtained the following result (1972).

Theorem

Let K be a field of characteristic $p > 0$ and assume that the group algebra $K[G]$ satisfies a polynomial identity of degree n. Then G has a normal subgroup A of index $\leq a(n)$ such that its commutator subgroup A' is a finite p-group of order $\leq b(n)$.

A group A whose commutator subgroup A' is a finite p-group is said to be p-abelian. The above result actually characterizes groups with IP_n for some n, in characteristic $p > 0$. Indeed, G is such a group if and only if it has a p-abelian subgroup of finite index.

The Permutational Property P_n

Following Curzio, Longobardi, Maj and Robinson (1985), a group G is said to have the permutational property P_n if for all $x_1, x_2, \ldots, x_n \in G$, there exists a nonidentity permutation $\pi \in \mathrm{Sym}_n$ (depending on these elements) with $x_1 x_2 \cdots x_n = x_{\pi(1)} x_{\pi(2)} \cdots x_{\pi(n)}$.

Proposition

If G satisfies IP_n for any field K, then it satisfies P_n.

Indeed, suppose $K[G]$ satisfies a polynomial identity of degree n. Then, via linearization, $K[G]$ satisfies a multilinear polynomial f of the form

$$f(\zeta_1, \zeta_2, \ldots, \zeta_n) = \sum_{\sigma \in \mathrm{Sym}_n} k_\sigma \zeta_{\sigma(1)} \zeta_{\sigma(2)} \cdots \zeta_{\sigma(n)}$$

with coefficient $0 \neq k_1 \in K$. Now note that $f(x_1, x_2, \ldots, x_n) = 0$, so the identity term in f must be cancelled by suitable $1 \neq \pi$ terms.

The Finite Conjugate Center

Let $\Delta(G)$ be the set of elements of group G having finitely many G-conjugates. This is the F. C. center of G. It is a characteristic subgroup. The result of [CLMR] asserts

Theorem

If G satisfies P_n, then $|G : \Delta(G)| \leq a(n)$ and $\Delta(G)'$ is finite.

The latter is the best they can do because $|\Delta(G)'|$ is not bounded by a function of n. There are even easy finite examples.

Lemma

Let $N \subseteq G$. If $|G : N| \leq a$ and $|N'| \leq b$ then G satisfies P_{2ab}.

Note that such a subgroup N is in $\Delta(G)$. One should really look for a converse of this and not just with $N = \Delta(G)$.

Classes of Bounded Size

It seemed that the old PI techniques could prove this converse, but that the task should be saved for a student. We list some of these methods.

Let $\Delta_k(G)$ be the set of all elements of G having $\leq k$ conjugates. Note that $\Delta_r(G)\Delta_s(G) \subseteq \Delta_{rs}(G)$ and $\Delta_r(G)^{-1} = \Delta_r(G)$. Of course these subsets are not necessarily subgroups. The following was proved by Wiegold (1957).

Theorem

Let G be a group and let k be an integer.

1. *If $|G'| \leq k$, then $G = \Delta_k(G)$.*
2. *If $G = \Delta_k(G)$, then $|G'| \leq (k^4)^{k^4}$.*

Part (2) above was a conjecture of B. H. Neumann (1954).

Subsets of Finite Index

Since $\Delta_r(G)$ is not a subgroup, one has to deal with subsets of G. We say a subset T of G has index $\leq k$ if there exist group elements $x_1, x_2, \ldots, x_k$ with $\bigcup_1^k T x_i = G$. Obviously this is not right-left symmetric. Write $T^* = T \cup 1 \cup T^{-1}$.

Lemma

If $|G : T| \leq k$, then $(T^*)^{4^k}$ is a subgroup of G.

Lemma

Suppose $H_1, H_2, \ldots, H_k$ are subgroups of G and set $S = \bigcup_1^k H_i x_i$.

1. *If $S = G$, then $|G : H_i| \leq k$ for some i.*
2. *If $S \neq G$. then there exist g_j for $1 \leq j \leq (k+1)!$ with $\bigcap_j S g_j = \emptyset$. In particular, if $S \cup T = G$, then $|G : T| \leq (k+1)!$.*

Characterization of P_n-Groups

This and some later work is joint with my student Mustafa Elashiry (2011).

Theorem

Let G be a group satisfying the permutational property P_n and set $k = n!$. Then we have

1. $|G : \Delta_k(G)| \leq k{\cdot}(k+1)!$, *and*
2. *G has a characteristic subgroup $N = \langle \Delta_k \rangle$ with $|G : N| \leq k{\cdot}(k+1)!$ and with $|N'|$ finite and bounded by a function of n.*

The latter bound is big. Set $l = k{\cdot}(k+1)!$. Then $N = (\Delta_k(G))^{4^l} \subseteq \Delta_m(G)$ where $m = k^{4^l}$. So $N = \Delta_m(N)$ and hence $|N'| \leq (m^4)^{m^4}$.

The Rewritable Property Q_n

Following R. D. Blyth (1988), we say that a group G satisfies the rewritable property Q_n if for all $x_1, x_2, \ldots, x_n \in G$ there exist distinct permutations $\sigma, \tau \in \mathrm{Sym}_n$, depending on these elements, with $x_{\sigma(1)} x_{\sigma(2)} \cdots x_{\sigma(n)} = x_{\tau(1)} x_{\tau(2)} \cdots x_{\tau(n)}$. Obviously

Lemma

If G satisfies P_n, then it satisfies Q_n.

Lemma

If $|G'| < n!$, then G satisfies Q_n.

Recall, if $|G'| \leq n/2$ then G satisfies P_n. Are these properties the same or just similar?

Examples and Blyth's Theorem

$G = \mathrm{Sym}_3$ satisfies Q_3 but not P_3. Q_3 follows from the previous lemma. For P_3, notice that the product $(1\,2\,3)\cdot(2\,3)\cdot(1\,3\,2) = (1\,2)$ is not equal to any other permuted product. Blyth has a generalization of this with G_n a cyclic group of odd order acted on by a cyclic 2-group. These groups have property Q_n but not P_n for all $n \geq 3$.

Note that the previous lemma implies that $G = \mathrm{Sym}_n$ satisfies Q_n. It does not satisfy Q_{n-1} by considering the $(n-1)$-fold products of the form $(1\,2)\cdot(1\,3)\cdot(1\,4)\cdots(1\,n) = (1\,2\,3\,4\cdots n)$.

Theorem

If G satisfies Q_n, then $|G : \Delta(G)| \leq a(n)$ and $\Delta(G)'$ is finite.

Obviously this is similar to the P_n result. But the proof is surprisingly much more difficult and uses a really neat trick. Fortunately, Blyth's trick can be merged in with the old PI techniques to yield:

Characterization of Q_n-Groups

Theorem

Let G be a group satisfying the rewritable property Q_n. Then there exist functions k, l and m of n with

1. $|G : \Delta_k(G)| \le l$, *and*
2. *G has a characteristic subgroup $N = \langle \Delta_k \rangle$ with $|G : N| \le l$ and with $|N'| \le m$.*

Corollary

If G is a group satisfying the rewritable property Q_n, then G satisfies the permutational property P_c for some function c of n.

The bounds here are big. For example, k, l and c are determined via

$$j = n!, \quad p = j^2, \quad q = p \cdot 2^p, \quad k = j \cdot q^p, \quad l = k \cdot (k+1)!, \quad c = 2ml$$