ITERATED STRONG TILTING AND STRINGS OF DERIVED-EQUIVALENT ALGEBRAS

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ABSTRACT. We review existing work on the connection between tilting and derived equivalences of finite dimensional algebras. Then we will recall the notion of *strong* tilting, as introduced by Auslander and Reiten. Strong tilting provides particularly effective contravariant bridges among the module categories involved. We proceed to collaborations of the speaker with A. Dugas and M. Saorín. The main results show: Any truncated path algebra has a strong tilting module. Moreover, if anchored in a truncated path algebra, the process of strong tilting allows for iteration and eventually becomes stationary. The algebras in the corresponding sequence of consecutive tilts can be described with precision in terms of their predecessors.

Let $\Lambda = KQ/I$ be a path algebra modulo relations, where Q is a quiver and K a field. Moreover, set $n = \operatorname{rank} K_0(\Lambda)$; i.e., Q has n vertices, say $e_1, \dots e_n$, corresponding to the isomorphism classes $S_1, \dots, S_n$ of simple modules.

The main point of my lecture is to discuss algebras which result from a given one by way of what is called "strong tilting". The plan is to study this strong form of tilting and to advance the homological understanding of truncated path algebras in tandem. But before I go into any detail, I'll try to place the specific problems I'll address in a somewhat broader perspective.

I. ENVIRONMENT

It has become clear that, in exploring algebras of wild representation type, the most promising tack is to mix different techniques and to tesselate the partial pictures that arise from the various lines of approach. Let me remind you of some of the mainstays among the techniques towards this end:

• Geometric methods. They only serve as motivation in this lecture, to be re-encountered at the end.

• Homological methods. One strategy along this line is to approximate a given representation of Λ by a member of a more thoroughly understood class of representations. Today,

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I'll address "best approximations" of arbitrary modules by modules of finite projective dimension, i.e., by modules in the following subcategory of Λ-mod:

$$\mathcal{P}^{<\infty} = \mathcal{P}^{<\infty}(\Lambda\text{-mod}) = \{X \in \Lambda\text{-mod} \mid \operatorname{p\,dim} X < \infty\}.$$

Best approximations, in a sense which I'll shortly make precise, follow the model of projective covers. But typically, one has a much larger class of approximating objects at one's disposal than just the projective modules, and hence one may reasonably expect a closer fit of the approximating objects to the modules on which one tries to zero in.

• Functorial methods. They are closely connected to homological methods. First in line, there are the Morita equivalences and Morita dualities which have been used since our relative representation-theoretic antiquity. For algebras that exhibit similar features, but fail to be Morita equivalent or dual, partial equivalences (co- or contravariant) have proved effective towards shifting information from one module category to another. By a "partial equivalence" I mean a functor which induces equivalences on certain subcategories of the module categories involved.

Presently, the focus is on tilting functors. Just like Morita equivalences, they are induced by bimodules over the two algebras that are being compared. But in contrast to Morita theory, the technique of tilting evolved in dozens of incremental steps. It was triggered by the success encountered by Bernstein-Gelfand-Ponomarev in the early 70's, in comparing the path algebra of a given acyclic quiver with that of a quiver obtained by reversing certain arrows. The resulting functors between the module categories of the corresponding path algebras now go by the name of *Bernstein-Gelfand-Ponomarev reflection functors*. Originally, they served the purpose of simplifying the proof of Gabriel's theorem on the representation type of a path algebra. Subsequently, the underlying idea was raised to a more abstract level and went through many stages of generalization. First, it was picked up by Auslander-Platzeck-Reiten, then a major leap occurred in papers by Brenner-Butler and Happel-Ringel; the arguments were smoothed out by Bongartz. But these first generations of the theory exclusively addressed tilting modules of projective dimension at most 1. The departure from this restriction was ushered in by Miyashita who broadened the method to tilting objects of arbitrary finite projective dimension. At this point too many players entered the game to make it reasonable to continue a systematic list.

However, the deeper reasons for the strong links among certain subcategories of representations over two algebras which are connected via a tilting bimodule was recognized only about 15 years after the Bernstein-Gelfand-Ponomarev reflection functors had made their entrance. Namely, Happel observed that, whenever two algebras are tied together by a tilting bimodule, their derived categories are triangle equivalent. This observation, in turn, led to a succession of generalizations, first by Cline-Parshall-Scott, then, most notably by Rickard and Keller. As Rickard first proved, two algebras, say Λ and $\widetilde{\Lambda}$, are derived equivalent if and only if there exists a tilting object in the derived category of Λ such that $\widetilde{\Lambda}$ is isomorphic to the endomorphism ring of this object.

The derived development gave tilting theory another big push: For one thing, Beilinson and Bernstein-Gelfand-Gelfand showed that the category of coherent sheaves over projective space is derived equivalent to the module category of a finite dimensional algebra.

Another reason for the burst of derived activity in the finite dimensional community lies in the fact that derived equivalences found immediate applications to group algebras. In particular, Broué formulated a conjecture that relates blocks of finite dimensional group algebras with abelian defect groups to their Brauer correspondents by way of derived equivalences. This conjecture subsumes and unifies a string of previous conjectures.

II. What is tilting?

Definition. 1. $T \in \Lambda$-mod is called a *tilting module* if

(i) $\operatorname{p\,dim} T < \infty$;

(ii) $\operatorname{Ext}^i_\Lambda(T,T) = 0$ for all $i > 0$;

(iii) The projectives in Λ-mod have finite right resolutions by objects in $\operatorname{add}(T) := \{X \mid X \subseteq^{\oplus} T^r, r \in \mathbb{N}\}$. More precisely, there exists an exact sequence

$$0 \to \Lambda \to T_1 \to \cdots \to T_m \to 0$$

for some m such that all T_j belong to $\operatorname{add}(T)$.

Comment: The number of isomorphism classes of indecomposable direct summands of any tilting module T over Λ is equal to the rank n of the Grothendieck group of Λ.

2. Call a tilting module ${}_\Lambda T$ *basic* if no indecomposable direct summand of T occurs with a multiplicity > 1, i.e., T has precisely n pairwise non-isomorphic indecomposable direct summands.

Suppose T is a tilting module. **Comparison algebra**: $\widetilde{\Lambda} = (\operatorname{End}_\Lambda(T))^{\mathrm{op}}$. Then ${}_\Lambda T_{\widetilde{\Lambda}}$ is obviously a bimodule. Not so obviously, the situation is symmetric in Λ and $\widetilde{\Lambda}$, meaning that $T_{\widetilde{\Lambda}}$ is a tilting module in mod-$\widetilde{\Lambda}$ and $\Lambda \cong \operatorname{End}_{\widetilde{\Lambda}}(T)$.

The **nub of the matter**: If ${}_\Lambda T_{\widetilde{\Lambda}}$ is a tilting bimodule, the functors $\operatorname{Hom}_\Lambda(T,-)$, $M \otimes_{\widetilde{\Lambda}} -$ and their derived functors provide equivalences when restricted to suitable subcategories of Λ-mod and $\widetilde{\Lambda}$-mod. (A familiar example: $T \in \Lambda$-mod is tilting if T is a projective generator. In that case, $\widetilde{\Lambda}$ is Morita equivalent to Λ.) In addition, there are weak clones of Morita *dualities* resulting from a tilting bimodule ${}_\Lambda T_{\widetilde{\Lambda}}$, which were first discovered by Miyashita. Such dualities are far less thoroughly explored than the covariant equivalences that link up portions of the two module categories involved. I will place primary emphasis on them in my lecture.

As already mentioned: If Λ and $\widetilde{\Lambda}$ result from each other via tilting, the derived categories $D^b(\Lambda\text{-mod})$ and $D^b(\widetilde{\Lambda}\text{-mod})$ are equivalent as triangulated categories. In particular, all derived invariants are tilting invariants.

Invariants under derived equivalence [Rickard, Happel, Keller, Rouquier, HZ-Saorín]: Not surprisingly, many homological invariants of Λ are preserved by derived equivalence, such as cyclic homology and Hochschild cohomology. Moreover, finiteness of global and finitistic dimensions goes through; however, the exact values of these dimensions do not. Further derived invariants are $K_0(\Lambda)$, as well as the center and the identity component of the outer automorphism group of Λ.

If Λ and $\widetilde{\Lambda}$ are tilting equivalent, one may obtain explicit bounds relating the homological dimensions of Λ to those of $\widetilde{\Lambda}$ in terms of the tilting object.

III. Strong tilting and contravariant finiteness

Typically, there is a plethora of tilting modules over a given finite dimensional algebra, and representation theorists have tried to introduce some order into the zoo. There are distinguished specimens (beyond the projective generators which induce Morita equivalences) which were first put under a spotlight by Auslander and Reiten who dubbed them *strong tilting modules*. Coming from a different viewpoint, Happel and Unger again encountered the strong tilting modules as noteworthy objects: Namely, Happel-Unger equipped the full class of basic tilting modules over Λ with a natural partial order and noticed that, existence provided, a basic strong tilting module is the smallest element of this poset. (By way of comparison: ${}_\Lambda\Lambda$ is the largest element.)

Strong tilting. A tilting module ${}_\Lambda T$ is *strong*, if it is a relatively injective object in the category $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$, i.e., $\text{Ext}^i_\Lambda(-,T)|_{\mathcal{P}^{<\infty}} = 0$.

Comment: In case of existence, there is a *unique basic strong tilting module* over Λ.

A first example: If Λ has finite global dimension, then the minimal injective cogenerator $T \in \Lambda\text{-mod}$ is the basic strong tilting module for Λ. As is well known, such a tilting module T induces the standard duality $\text{Hom}_\Lambda(-,T) \cong \text{Hom}_K(-,K) : \ \Lambda\text{-mod} \to \text{mod-}\widetilde{\Lambda}$.

To see the parallel with more general instances of strong tilting, observe: If T is any strong tilting module, then T is an injective cogenerator for the category $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$, and each object in $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$ has finite relative Ext-injective dimension. In other words, the classical setting is reproduced for the subcategory $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$ of Λ-mod.

The status quo of the "strong" theory: The strengthened connections between algebras which are equivalent under *strong tilting* have not been fully exposed yet as far as I can see. In fact, the main focus has been on *covariant* equivalences among subcategories of module categories; tilting theory emerged as a generalization of Morita equivalence after all. On closer inspection it turns out that *dualities* are the bridges that most obviously gain prominence under *strong* tilting. Another retarding factor lies in the sparsity of nontrivial examples so far. The reason for the small size of the playground lies in the fact that strong tilting modules don't always exist and that it is typically nontrivial to verify (or refute) existence.

Here is the punch line regarding existence: Auslander-Reiten proved that Λ-mod has a strong tilting module iff $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$ is contravariantly finite.

And here is what this means:

Contravariant finiteness. [Auslander-Smalø, Enochs]

1. $M \in \Lambda\text{-mod}$ is said to have a (right) $\mathcal{P}^{<\infty}$*-approximation* if there exists a homomorphism $\phi : A \to M$ such that $\mathcal{A} \in \mathcal{P}^{<\infty}(\Lambda\text{-mod})$ and every map in $\text{Hom}_\Lambda(\mathcal{P}^{<\infty}, M)$ factors through ϕ.

If M has *some* $\mathcal{P}^{<\infty}$-approximation, then there exists a *minimal* one $\mathcal{A}(M)$ say, which is unique up to isomorphism.

2. In case all left Λ-modules have $\mathcal{P}^{<\infty}$-approximations, we say that $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$ is *contravariantly finite*.

Clearly, $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$ is contravariantly finite whenever $\operatorname{gldim}\Lambda < \infty$; in that case, the minimal approximation of any Λ-module M is M itself (the identity map on M, more precisely). On the other extreme, if the left finitistic dimension of Λ is 0, then $\mathcal{P}^{<\infty}$ is in turn contravariantly finite; in that case, the minimal approximation of a module is just its projective cover. However, situations where this concept really has traction lie strictly in between these extremes.

Benefits of contravariant finiteness of $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$:

The importance of the property is due to the following facts: (a) The category $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$ has internal Auslander-Reiten sequences in case it is contravariantly finite, and (b) contravariant finiteness of $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$ is a marvel of a situation from a homological viewpoint: Indeed, if $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$ is contravariantly finite, then the modules of finite projective dimension can be characterized in terms of finitely many building blocks, namely the minimal approximations of the simple modules. Moreover, the big and little finitistic dimensions coincide and are finite in this case. Arbitrary modules of finite projective dimension are simply direct limits of finitely generated modules of finite projective dimension, etc. (c) Most notably in the present context [Auslander-Reiten]:

$$\Lambda\text{-mod has a strong tilting module} \iff \mathcal{P}^{<\infty}(\Lambda\text{-mod}) \text{ is contravariantly finite.}$$

The algebras with contravariantly finite $\mathcal{P}^{<\infty}$-categories are abundant; but the condition is tough to verify or refute. I quote a remark from a paper of Auslander and Reiten (1991): "Little is known about the general question of when the subcategory of modules of finite projective dimension is contravariantly finite." The first example for failure of this condition was given by Igusa and Todorov, still in the 90's. The situation has improved in the meantime, but not decisively enough to anchor the theory in a solid environment to which it applies. In particular, there are hardly any major classes of algebras all of which have contravariantly finite $\mathcal{P}^{<\infty}$-categories. It is a bit as with transcendental numbers: They are everywhere dense, but it's nontrivial to pin down specific ones.

So, with regard to strong tilting, the upshot of the discussion is this: What one needs for a better understanding of this mode of comparison of two algebras is a broader class of algebras Λ for which contravariant finiteness of $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$ is confirmed and a structural description of the corresponding strong tilting modules is available.

IV. Specific goals to be addressed in the following

• Improve the understanding of strong tilting.

Our starting point is essentially due to Miyashita. If ${}_\Lambda T_{\widetilde{\Lambda}}$ is a tilting module which is strong on both sides, then the functors $\operatorname{Hom}_\Lambda(-,T)$ and $\operatorname{Hom}_{\widetilde{\Lambda}}(-,T)$ induce inverse dualities

$$\mathcal{P}^{<\infty}(\Lambda\text{-mod}) \longleftrightarrow \mathcal{P}^{<\infty}(\text{mod-}\widetilde{\Lambda}).$$

• Advance the theory of truncated path algebras (i.e., of algebras KQ/I, where I is generated by all paths of some fixed length) to a level matching that of split hereditary finite dimensional algebras (i.e., of path algebras KQ, where Q is an acyclic quiver).

Clearly, every path algebra of an acyclic quiver is a truncated path algebra. In fact, the role played by truncated path algebras with respect to arbitrary path algebras modulo relations parallels the role played by the hereditary algebras relative to algebras whose Gabriel quivers contain no oriented cycles.

Namely: If $\Lambda = KQ/I$ for an arbitrary quiver Q and admissible ideal $I \subseteq KQ$, then Λ is a quotient of a unique truncated path algebra Δ that has quiver Q and the same Loewy length as Λ. The incentive for following up on this analogy: In studying a Λ-module M, it turns out to be very helpful to use the embedding $\Lambda\text{-Mod} \hookrightarrow \Delta\text{-Mod}$ and move back and forth between the Λ- and Δ-structures of M.

Starting point [Dugas-HZ]: For any truncated path algebra Λ, the category $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$ is contravariantly finite.

On the other hand, there are plenty of differences between truncated algebras and hereditary algebras. So the theory of truncated path algebras (which subsumes that of hereditary algebras) is necessarily more complex. This fact already surfaced in the geometric setting and is equally pronounced on the homological side of the picture. For instance, the global dimension of a truncated path algebra is infinite whenever Q has oriented cycles, and arbitrary discrepancies between the left and right finitistic dimensions can be realized within this class of algebras.

V. Strongly tilting truncated path algebras

From now on

$$\Lambda = KQ/\langle \text{all paths of length } L+1 \rangle, \quad L \geq 1 \text{ fixed.}$$

In this situation, Λ-Mod, the category of left Λ-modules, depends only on Q and the Loewy length $L+1$ of Λ. Hence it is to be expected that the combinatorics of the quiver should play a prominent role in any analysis of this category. In fact, a mere glance at the quiver Q allows to pinpoint the vertices that give rise to the simple left Λ-modules of finite projective dimension; what is decisive is the placement of the corresponding vertices relative to the oriented cycles.

A vertex e_i of Q is called *precyclic* if there is a path in Q which starts in e_i and ends on an oriented cycle; dually, e_i is *postcyclic* in case there is a path that starts on an oriented cycle and ends in e_i. Finally, we call e_i *critical* if e_i is both pre- and postcyclic. We extend this terminology to the indecomposable projective and simple left Λ-modules that correspond to e_i.

It is a cinch to verify the following: A simple left Λ-module has finite projective dimension if and only if it is non-precyclic.

First an old result of Dugas-HZ.

Theorem A. *Suppose Λ is a truncated path algebra. Then $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$ is contravariantly finite, and the minimal $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$-approximations of the simple modules are known personally. Moreover, there is an explicit description of the basic strong tilting module T. In particular, T is constructible from Q and L. The corresponding strongly tilted algebra $\widetilde{\Lambda} = K\widetilde{Q}/\widetilde{I}$ can in turn be determined from these data.*

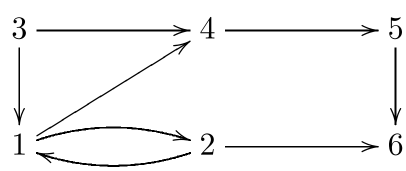
Remark and convention. We have $K_0(\Lambda) \cong \mathrm{K}_0(\widetilde{\Lambda})$, whence the quiver $\widetilde{Q}$ has the same number of vertices as Q, say $\widetilde{e}_1, \dots, \widetilde{e}_n$. It turns out that there is a canonical correspondence between the vertices of Q and those of $\widetilde{Q}$. (I omit detail.) On referring to the $\widetilde{e}_i$, I will tacitly assume that the order of the lineup reflects this correspondence, so as to make the following unambiguous: An idempotent $\widetilde{e}_i \in \widetilde{\Lambda}$ is a *critical* vertex of $\widetilde{Q}$ if e_i is critical in Q; in that case, the projective $\widetilde{e}_i\widetilde{\Lambda}$ and its simple quotient are also called critical. Caveat: This convention is not in agreement with the role played by the $\widetilde{e}_i$ relative to the quiver $\widetilde{Q}$; the latter quiver teems with oriented cycles in general (see below).

Definition. • The idempotent of $\widetilde{\Lambda}$ which plays the key role in the homological discussion of Mod-$\widetilde{\Lambda}$ is

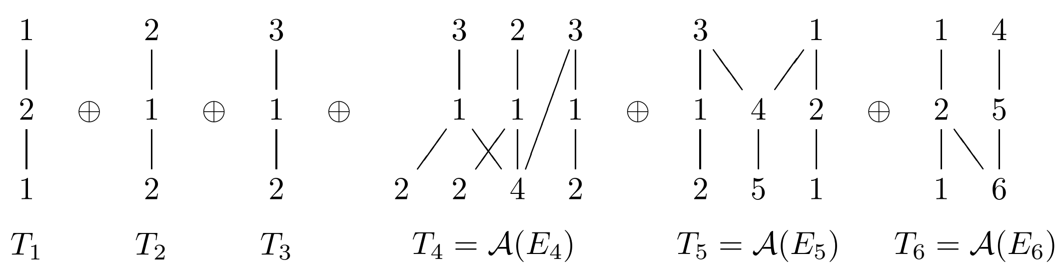
$$\widetilde{\mu} = \sum_{e_i \text{ critical}} \widetilde{e}_i.$$

• The *critical core* of $\widetilde{M} \in$ Mod-$\widetilde{\Lambda}$ is the subfactor V/U, where $V = \widetilde{M}\widetilde{\mu}\widetilde{\Lambda}$ and U is the annihilator in $\widetilde{M}$ of the left ideal $\widetilde{\Lambda}\widetilde{\mu}$. In particular, $\mathrm{top}(V/U)\widetilde{\mu} = \mathrm{top}(V/U)$ and $\mathrm{soc}(V/U)\widetilde{\mu} = \mathrm{soc}(V/U)$; if $\dim \widetilde{M} < \infty$, then V/U is, in fact, the subfactor of maximal dimension of $\widetilde{M}$ with the property that all simple summands of $\mathrm{top}(V/U) \oplus \mathrm{soc}(V/U)$ are critical.

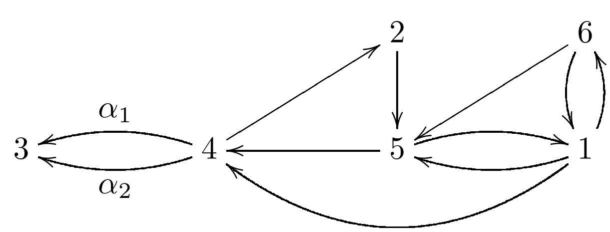
Example. Consider $\Lambda = KQ/\langle \text{all paths of length } 3\rangle$, where Q is



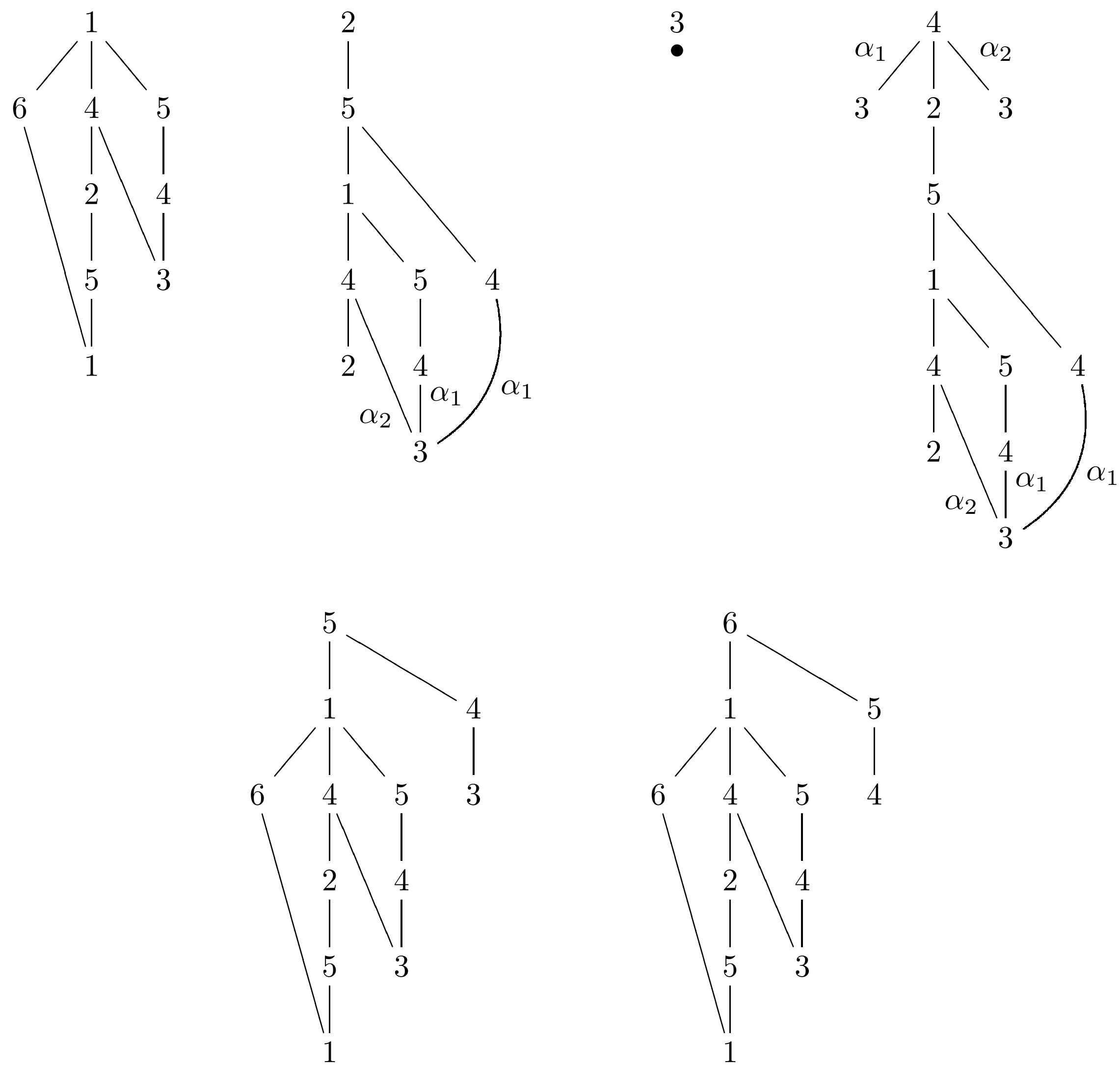
Clearly, e_1, e_2 are the only critical vertices of Q. The strong basic tilting module in $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$ is $T = \bigoplus_{i-1}^n T_i$, as pinned down by the following tree graphs:



As before, $\widetilde{\Lambda} = \mathrm{End}_\Lambda(T)^{\mathrm{op}}$ = strong tilt of Λ. The quiver of $\mathrm{End}_\Lambda(T)$:



The indecomposable projective right $\widetilde{\Lambda}$-modules $\widetilde{e}_i\widetilde{\Lambda}$:



The critical cores of the $\widetilde{e}_i\widetilde{\Lambda}$ are easy to identify; e.g., that of $\widetilde{e}_1\widetilde{\Lambda}$ is the quotient of $\widetilde{e}_1\widetilde{\Lambda}$ by the uniserial submodule with composition factors S_5, S_4, S_3.

Observe that the analogies between hereditary algebras and truncated path algebras emerge more clearly on the level of strong tilts. Indeed, whenever an indecomposable projective right $\widetilde{\Lambda}$-module $\widetilde{P}$ has a critical composition factor, then $\widetilde{P}$ contains a copy of one of the critical indecomposable projectives $\widetilde{e}_1\widetilde{\Lambda}$ and $\widetilde{e}_2\widetilde{\Lambda}$; in fact, all critical composition factors of $\widetilde{\Lambda}_{\widetilde{\Lambda}}$ are confined to such copies of the critical projectives. This is not just a fluke, but is true in general.

The obvious next step is to settle the following

Question: Is the tilting module $T_{\widetilde{\Lambda}}$ strong also in mod-$\widetilde{\Lambda}$?

Answer: No in general. In fact, Dugas and I showed that the answer is positive precisely when all precyclic vertices of Q are also postcyclic. This partial answer begs a

Followup Question: Namely, does mod-$\widetilde{\Lambda}$ have its own strong tilting module, i.e., is $\mathcal{P}^{<\infty}(\text{mod-}\widetilde{\Lambda})$ always contravariantly finite even when $T_{\widetilde{\Lambda}}$ fails to be a strong tilting module?

Recently, Saorín and I settled the issue affirmatively. To make headway, one requires a thorough understanding of the $\widetilde{\Lambda}$-modules of finite projective dimension.

Let me start with the simple right $\widetilde{\Lambda}$-modules $\widetilde{S}_i = \widetilde{e}_i\widetilde{\Lambda}/\widetilde{e}_i\widetilde{J}$.

Theorem B. $\operatorname{p\,dim} \widetilde{S}_i < \infty$ *precisely when* $\widetilde{e}_i$ *is noncritical.*

This fact may be seen as a first indication that the homological picture is symmetrized on passage from Λ to $\widetilde{\Lambda}$. From Theorem B we are led to the following characterization of the right modules of finite projective dimension over $\widetilde{\Lambda}$.

Theorem C. *For* $\widetilde{M} \in \text{Mod-}\widetilde{\Lambda}$*, the following are equivalent:*

- $\operatorname{p\,dim} \widetilde{M}_{\widetilde{\Lambda}} < \infty$.
- $\widetilde{M}\widetilde{\mu}$ *is projective as a right* $\widetilde{\mu}\widetilde{\Lambda}\widetilde{\mu}$*-module, i.e.,* $\widetilde{M}\widetilde{\mu} \cong \bigoplus_{e_i \text{ critical}} (\widetilde{e}_i\widetilde{\Lambda}\widetilde{\mu})^{\tau_i}$ *for suitable cardinals* τ_i.
- *The critical core of* $\widetilde{M}$ *is a direct sum of copies of the critical cores of the* $\widetilde{e}_i\widetilde{\Lambda}$.

Theorem C allows us to answer the question of contravariant finiteness of $\mathcal{P}^{<\infty}(\text{mod-}\widetilde{\Lambda})$ in the positive.

Theorem D. *The category* $\mathcal{P}^{<\infty}(\text{mod-}\widetilde{\Lambda})$ *is always contravariantly finite in* $\text{mod-}\widetilde{\Lambda}$.

Moreover, the minimal $\mathcal{P}^{<\infty}(\text{mod-}\widetilde{\Lambda})$*-approximations of the* $\widetilde{S}_i$ *and the basic strong tilting module* $\widetilde{T} \in \text{mod-}\widetilde{\Lambda}$ *can be determined from* $\widetilde{Q}$ *and* $\widetilde{I}$. (*There is a theoretical description which allows for construction of these modules.*)

So how does this game continue?

Let $\widetilde{\widetilde{\Lambda}} = \operatorname{End}_{\widetilde{\Lambda}}(\widetilde{T})$. Is the tilting bimodule ${}_{\widetilde{\widetilde{\Lambda}}}\widetilde{T}_{\widetilde{\Lambda}}$ strong on both sides? The answer provides the strongest evidence so far for my assertion that moving from Λ to $\widetilde{\Lambda}$ symmetrizes the original algebra from a homological viewpoint.

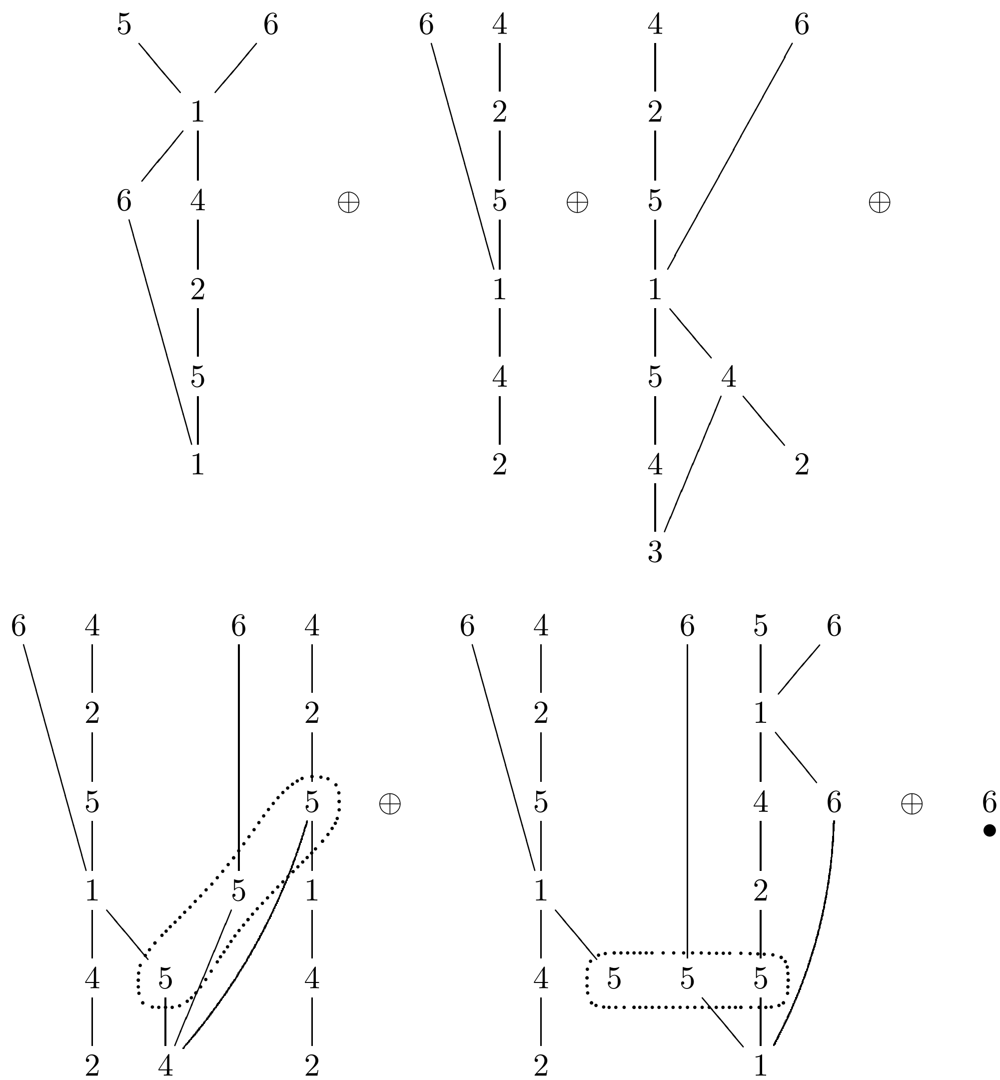
Theorem E. *YES.*
Indeed, ${}_{\widetilde{\widetilde{\Lambda}}}\widetilde{T}_{\widetilde{\Lambda}}$ *is a tilting module which is strong on both sides. In particular, the Hom-functors* $\operatorname{Hom}_{\widetilde{\Lambda}}(-,\widetilde{T})$ *and* $\operatorname{Hom}_{\widetilde{\widetilde{\Lambda}}}(-,T)$ *induce inverse dualities*

$$\mathcal{P}^{<\infty}(\text{mod-}\widetilde{\Lambda}) \leftrightarrow \mathcal{P}^{<\infty}(\widetilde{\widetilde{\Lambda}}\text{-mod}),$$

and $\widetilde{\widetilde{\widetilde{\Lambda}}} \cong \widetilde{\Lambda}$.

Final remark. To link up with a comment made at the outset: These homological connections should provide a good platform from which to explore those irreducible components of the module varieties of Λ, $\widetilde{\Lambda}$, $\widetilde{\widetilde{\Lambda}}$ which generically parametrize objects of finite projective dimension.

Return to the example. The basic strong tilting module $\widetilde{T}$ in $\mathcal{P}^{<\infty}(\text{mod-}\widetilde{\Lambda})$ has the following graph:



Here the dotted pools indicate linear dependencies.