

## Lines and the Cartesian Plane

### The Cartesian Plane

At this point, many people have seen several types of graphs. In the newspapers, one sees pie charts and bar charts. These are both forms of graphs. Often times in the stock market journals, one sees graphs that show the performance of stocks over time. These are all ways of attaching a visual representation to a bunch of numbers.

However, the sort of graph we are interested in here are graphs in the **cartesian plane**. The cartesian plane provides a pallate on which to graph, and it is built by taking two number lines and making them cross at a right angle where both of them are zero. Then, we have one number line going up and down, and another number line going from left to right. By convention, we make the positive numbers of the up and down number line go up, and the positive numbers of the left to right number line go to the right. The result is shown below in Fig 1.

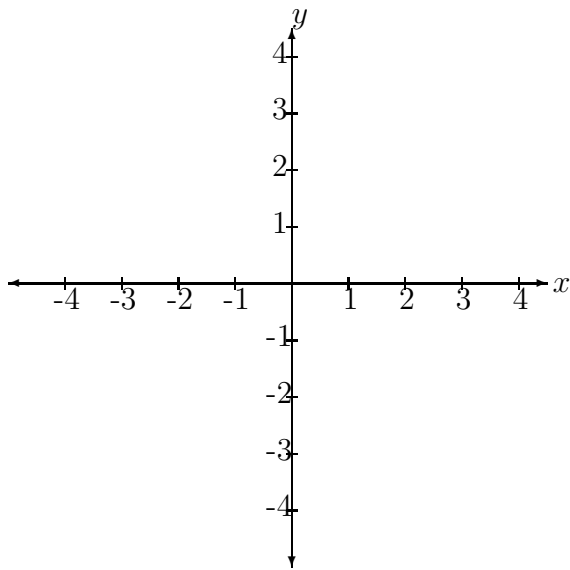


Figure 1: Cartesian Plane

If you take a long hard look at this, it is just two real number lines crossed

as described previously.

Notice that there is an  $x$  at the right end of the horizontal number line and a  $y$  at the top of the vertical number line. We call these two number lines *axes*. Specifically, we call the horizontal number line the  $x$ -*axis* and the vertical number line the  $y$ -*axis*. This is another convention mathematicians use. This gives us a way of identifying each point in the cartesian plane as an *ordered pair*  $(x, y)$ . It is a pair because there are two numbers, and it is ordered because we put the  $x$  value before the  $y$  value. For example, we can find  $(2, 3)$  in the cartesian plane by going along the  $x$  axis to 2 and along the  $y$  axis to 3. This is shown in Fig. 2.

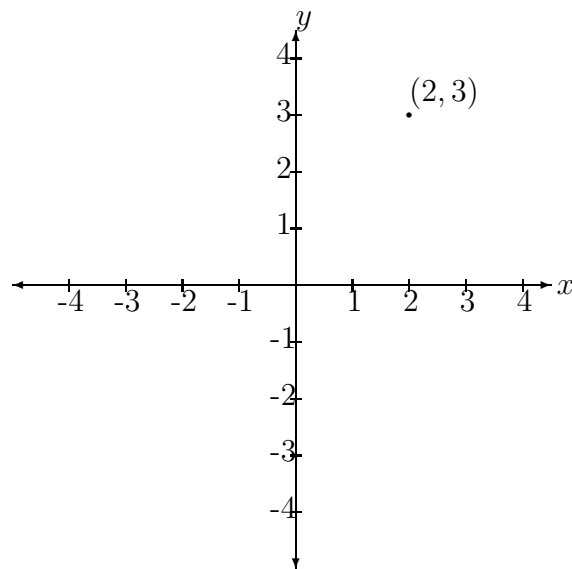


Figure 2: Ordered Pair

Now, we have some more terminology that goes along with the cartesian plane. Notice that there are four distinct areas of the cartesian plane. We name them *quadrants* because there are four of them. Each one has its own number, which we write as a roman numeral. That is, quadrants I (1), II (2), III (3) and IV (4). Their locations are indicated in the cartesian plane in Fig. 3.

Notice that in quadrants I and II,  $y$  is always positive. In quadrants I and IV,  $x$  is always positive. This pattern is illustrated in Fig. 4.

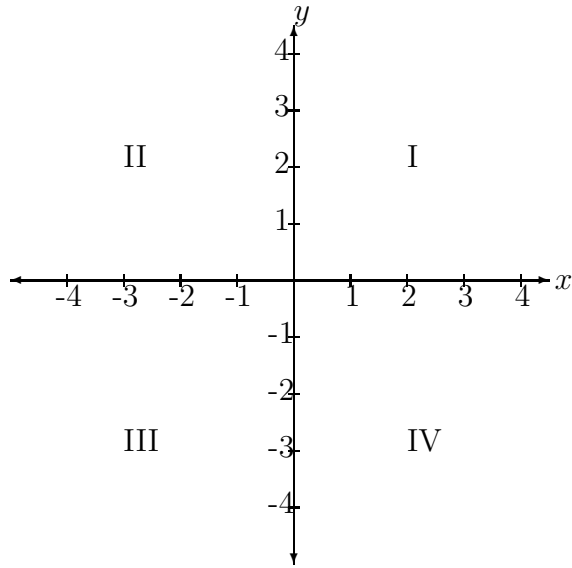


Figure 3: Quadrants of the Cartesian Plane

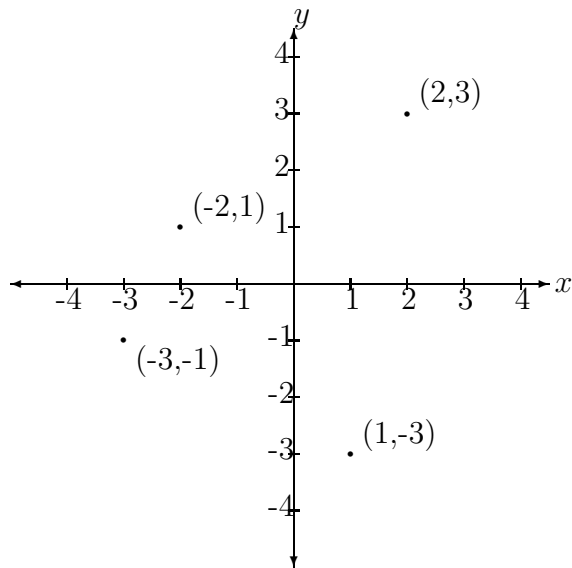


Figure 4: Ordered Pairs in each Quadrant

However, we don't have to pick integers as the  $x$  and  $y$  values to plot in the cartesian plane. In fact, since the plane is constructed with two real number lines, any real number will work for the  $x$  and  $y$  values. We will exploit this in later sections to plot more complicated things. Our next step is figuring out everything we can about lines.

## Lines

Many people have seen lines before, either in plots on the news or in other mathematics courses. However, if you have not, don't panic. We will go through all of the steps here.

Let's start by looking at an example to get an idea for what we are going to deal with. Let's look at  $y = 2x + 3$ . How do we start this? What are we supposed to do? What does this have to do with graphing? How is this a line?

The first question is the first one we should address. Since this has an equals sign ( $=$ ), this is an equation. We want values of  $x$  and  $y$  so that this equation is *true*. What do we mean by that? We mean that when we plug some  $x$  and  $y$  into the equation, we get something like  $5=5$ . If, however, we get something like  $4=5$ , the  $x$  and  $y$  we plugged in do not satisfy the equation. Therefore, if you happen to pick an  $x$  and a  $y$  value,  $(x, y)$ , and it gives you a true statement, you have found a point in the cartesian plane that satisfies the equation. We can then plot this point in the cartesian plane.

Our ultimate goal is to find *every* point  $(x, y)$  that satisfies this equation. But, we will take a step back now and devise a strategy for finding points that make this equation true because it is not easy to just pick a point that will work.

Since we have  $y = 2x + 3$ , and we want an  $x$  and a  $y$  so that this equation is true, we can pick any  $x$  and plug it in to the equation. Let's start simple, with  $x = 0$ . If we plug this into the equation, we get  $y = 2 \cdot 0 + 3 = 0 + 3 = 3$ , so  $y = 3$ . Then,  $(0, 3)$  is a point that satisfies this equation. We can clearly see that if we plug in 0 wherever we see  $x$  and plug in 3 wherever we see  $y$ , then we end up with  $3 = 3$ , which is a true expression.

Therefore, we can pick any  $x$  value we want, and plug it into the equation to solve for  $y$ . Then, the resulting  $y$  value with the  $x$  we plugged in gives us a point  $(x, y)$  that satisfies the equation. Let's try this for  $x = 1$  in the same equation  $y = 2x + 3$ . Plugging in 1 for  $x$  gives us  $y = 2 \cdot 1 + 3 = 2 + 3 = 5$ . So,

$y = 5$  when  $x = 1$ . Therefore, the point  $(1, 5)$  is a solution to this equation.

We can now plot these two points in the cartesian plane. This has been done in Figure 5. Notice that the line connecting the two points has been drawn also. This is because EVERY POINT on the line connecting these two points also satisfies the equation!

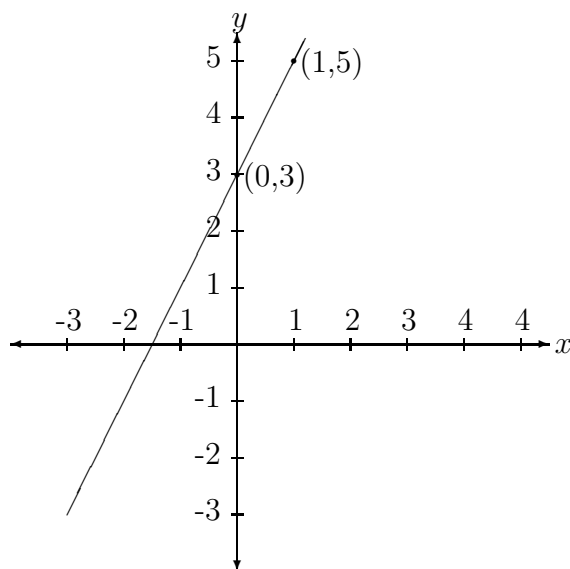


Figure 5:  $y = 2x + 3$

Therefore, the equation  $y = mx + b$  is the equation of a line in the cartesian plane. We have discussed how to draw a line in the cartesian plane if we are given the equation. We first pick a value for  $x$ , plug it into the equation, and then solve for  $y$ . This gives us one point  $(x, y)$  in the cartesian plane that satisfies the equation of the line. We then pick another value of  $x$ , and solve for a new value of  $y$  and get a second point satisfying the equation. Then, we draw the line connecting the points to get the graph of the line in the cartesian plane.

## All About $y = mx + b$

The equation  $y = mx + b$  is called the *slope-intercept* equation of a line. Why is this the slope-intercept equation? Are there other equations for a line?

These are two questions that we will answer in this section. But first, we need some definitions so we are all on the same page.

I called this the slope-intercept equation of a line, but never defined slope, or intercept. That isn't very fair. Let's start with the intercept. The intercept is the point(s) where the graph crosses one of the axes. There are two types of intercepts possible in the cartesian plane, so to avoid ambiguity we call the point(s) where the graph crosses the y-axis the *y-intercept* and the point(s) where the graph crosses the x-axis the *x-intercept*. The intercept referred to in the slope-intercept equation is the y-intercept.

Note that it is easy to calculate the y-intercept of a line that is not in the form  $y = mx + b$  just by setting  $x$  to zero and plugging it into the equation. This gives us the value of  $y$  when  $x = 0$ . But, the y-axis is at  $x = 0$ , so this value of  $y$  is the y-intercept. The same goes for finding the x-intercept. We plug  $y = 0$  into the equation and solve for  $x$ . This gives us the value(s) of  $x$  where the graph crosses the x-axis because  $y = 0$  everywhere on the x-axis.

Now, what about the slope? The slope tells you how steep the line is, and what direction it is going. Strictly speaking, there is an equation to calculate the slope if you have two points of the line,  $(x_1, y_1)$  and  $(x_2, y_2)$ . This is:

$$\text{slope } (m) = \frac{\text{rise}}{\text{run}} = \frac{\text{change in } y}{\text{change in } x} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

We have given 4 ways of writing the slope, starting with the intuitive definition and getting more technical at each step. All are equally correct. However, using "rise / run" can lead to sign mistakes, so the others are preferred.

Now, let's look at our initial equation of a line  $y = 2x + 3$ . We can calculate the slope because we know two of the points on the line,  $(0, 3)$  and  $(1, 5)$ . So, let's calculate the slope:

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{5 - 3}{1 - 0} = \frac{2}{1} = 2.$$

Therefore, the slope,  $m$ , of  $y = 2x + 3$  is 2. We can figure this out by "walking" along the imaginary grid created by the cartesian plane. And, according to our definition of slope as "rise/run," we go in the y-direction for "rise" units and then in the x-direction for "run" units. These steps are illustrated in Figure 6.

This graph (Fig. 6) illustrates an important point. We can walk along the graph by stepping  $\frac{-\text{rise}}{\text{run}}$  and get the same thing. This is because  $\frac{-1}{-1} = 1$ , and all we did was multiply the numerator and denominator by  $-1$ .

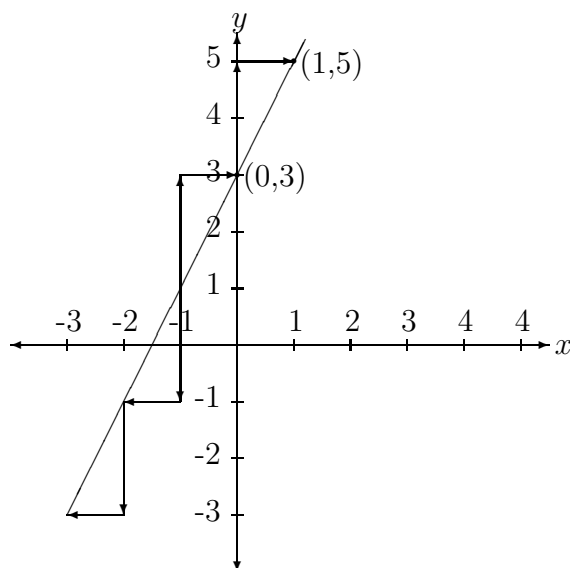


Figure 6: Slope as  $\frac{\text{rise}}{\text{run}}$

Now, looking at the graph in Fig. 5 or Fig. 6 we see that the line crosses the y-axis at  $y = 3$ , so our y-intercept,  $b$ , is 3. If we plug in  $m = 2$  and  $b = 3$  into  $y = mx + b$ , we see that we get  $y = 2x + 3$ , the equation we started with! This is why we call this the slope-intercept equation of a line! If we are given an equation (or can beat an equation into the form of)  $y = mx + b$ , where  $m$  and  $b$  are numbers, we know everything about the line!

“Not so fast!,” you say. “We only have one point (the y-intercept) and the slope! This is not enough information to determine the line!” According to our discussion this far, that is correct. However, consider the equation of the slope one more time. But, this time, we know the slope, and we can leave  $y_2$  and  $x_2$  as  $x$  and  $y$  which we want to find. Let’s use our slope of 2 that we had before and the point  $(0, 3)$  and see if we can re-create the equation of the line we had:

$$m = \frac{y_2 - y_1}{x_2 - x_1}, \text{ so}$$

$$2 = \frac{y - 3}{x - 0} = \frac{y - 3}{x}, \text{ multiply both sides by } x \text{ to get}$$

$2x = y - 3$ , and add 3 to both sides to get  $2x + 3 = y$ , or  $y = 2x + 3$ . So, starting with a point and a slope, we got back to the slope-intercept equation of the line! This means that if we have a slope and a point, we have all of the information we need to graph the line!

In general, the *point-slope* equation of a line is

$$(x - x_1) = m(y - y_1)$$

where we have the slope  $m$  and the point  $(x_1, y_1)$ .

Finally, there is a general form of a line which does not tell us any characteristics of the line, but can be manipulated to give us the equation of the line. The *general form* of a line is  $ax + by = c$ . To prove that this really is a line, consider  $4x - 2y = -6$ . This is not in the point-slope form or the slope-intercept form. But, we can try to make it into one of these forms. First, add  $2y$  to both sides. Then we get  $4x = 2y - 6$ . Then, add 6 to both sides to get  $4x + 6 = 2y$ . Now, divide both sides by 2 to get  $2x + 3 = y$ , or,  $y = 2x + 3$ , the same line we have been working with this whole time! However, in the general form, this equation didn't tell us anything. But, after solving for  $y$  (i.e. getting  $y$  to one side of the equation by itself), we get this into the slope-intercept form without much work.

## Slopes

There are several special ideas that I will mention here but not devote much explanation to. They all have to do with slopes.

The equation  $y = a$  where  $a$  is some number is a horizontal line that intersects the  $y$ -axis at  $a$ . This is because there is no  $x$  in the equation. Therefore, this is independent of the  $x$  you choose. So, for every  $x$ , the point  $(x, a)$  makes this equation true because there is no  $x$  in the equation.

The equation  $x = a$  is a vertical line, for the same reasons  $y = a$  is a horizontal line.

By definition, a vertical line and a horizontal line are perpendicular. This means that the angle between the two lines in the graph is 90 degrees. This idea of perpendicular lines can be extended to any two lines. If  $y = m_1x + b_1$  and  $y = m_2x + b_2$  are the equations of two lines, then they are perpendicular if  $m_1 \cdot m_2 = -1$ . This means that if we have any line  $y = mx + b$ , any line perpendicular to it has an equation like  $y = -\frac{1}{m}x + c$  because  $-\frac{1}{m} \cdot m = -1$ .

Finally, parallel lines are just two lines that have the same slope. If two lines have both the same slope and  $y$ -intercept, then they are the same line. However, if they have the same slope and a different intercept, then they will never touch each other.

## Line Segments

Line segments are just what they sound like: they are sections of a line. Typically, we define a line segment by giving its endpoints. For example, the line segment connecting  $(0, 3)$  and  $(-2, -1)$  is just the part of a line connecting these two points. This is shown in Fig. 7.

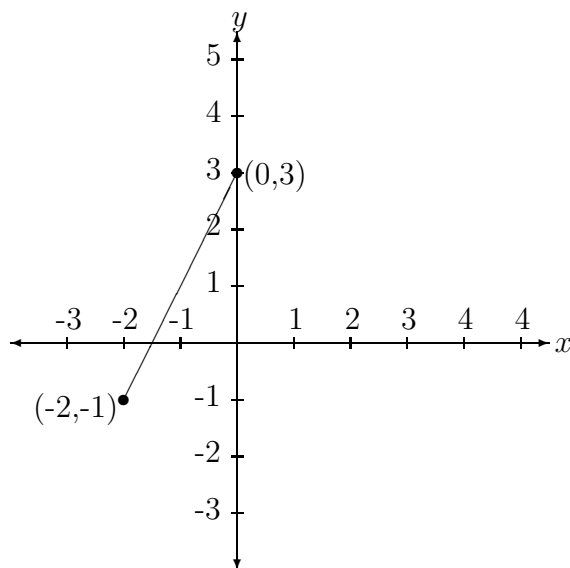


Figure 7: Line Segment

Is there a way to figure out the distance between these two points? Sure, you could buy a ruler and measure them, but then you are just getting an estimate. We want the exact length. Is that too much to ask? It turns out that it really isn't. Remember the Pythagorean Theorem that has to do with right triangles? Well, we can construct a right triangle around this segment to find its length. We can draw horizontal and vertical lines from the points to get a triangle with sides we can measure. In fact, sometimes we can even let the axes do the measuring for us!

In Figure 8, we have drawn the only line necessary to make a triangle. Now, let's look at the vertical side of the triangle. This has length  $|3 - (-1)| = |3 + 1| = 4$ . Looking at the vertical side, we see that the length is  $|-2 - 0| = |-2| = 2$ . Therefore, we can calculate the hypotenuse length by the equation  $a^2 + b^2 = c^2$ , and plugging in the numbers for the sides gives us

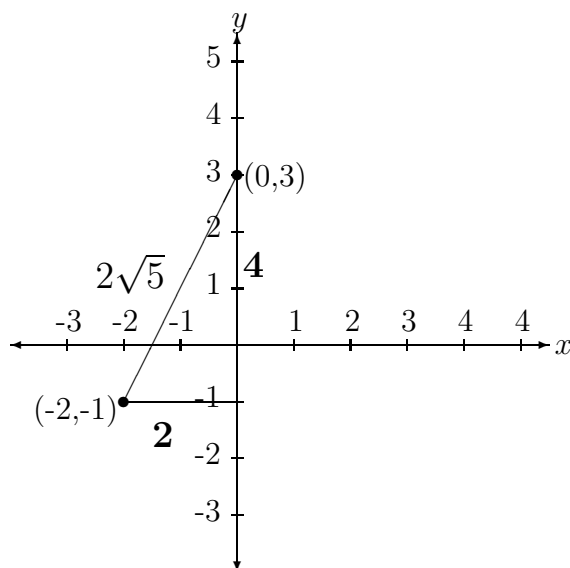


Figure 8: Line Segment Length

$c^2 = 2^2 + 4^2 = 4 + 16 = 20$ . So,  $c = \sqrt{20} = 2\sqrt{5}$ . Therefore, the length of our line segment is  $2\sqrt{5}$ .

This may seem like a trick, but it always works. The vertical side of the triangle will always have length  $|y_2 - y_1|$  and the horizontal side will always have length  $|x_2 - x_1|$ . So, we can plug these into the pythagorean equation for right triangles to get:

$$\text{distance} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

The keen reader would have noticed that we left out the absolute value! However, this is not a mistake since  $a^2 = (-a)^2$ . So, when we square the terms, they become positive, making the absolute value excessive.

The final question we may ask is: “what is the midpoint of this segment?” Well, if the numbers are nice and clean, we can usually tell by looking at the graph. However, people do not always draw perfect graphs. Therefore, it is necessary to find an equation to calculate it exactly.

So, we have two points. We want to know the midpoint of the segment connecting them. If we can calculate where “half way” between the x values is and where “half way” between the y values is, then we have found “half way” along the segment. This is just taking the average (mean), of the x

values to get the new  $x$  value, and the same for  $y$ . So, if you have two points,  $(x_1, y_1)$  and  $(x_2, y_2)$ , the midpoint of the segment connecting the two is given by:

$$\text{midpoint} = \left( \frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right)$$

This is nothing but the average value of  $x$  and the average value of  $y$  on the segment. For the segment we have been considering, we have endpoints  $(0, 3)$  and  $(-2, -1)$ . Putting these into the equation gives us a midpoint of  $\left( \frac{0+(-2)}{2}, \frac{3+(-1)}{2} \right) = \left( \frac{-2}{2}, \frac{2}{2} \right) = (-1, 1)$ . This has been plotted in Figure 9

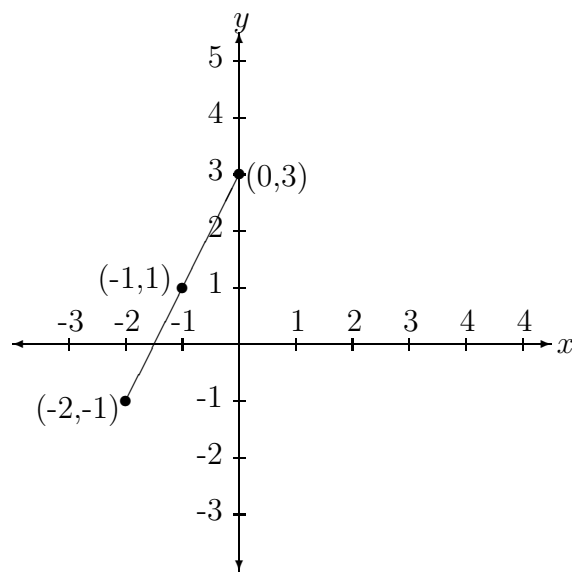


Figure 9: Midpoint