Higher regularity of uniform local minimzers in Calculus of Variations

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Abstract

A simple link between minimization property and H^2 regularity is found for Lipschitz extremals of vectorial variational problems.

1 Introduction

For vectorial variational problems the regularity question does not have a simple answer. For strictly convex variational problems there are regularity theorems when the number of independent variables does not exceed two, [11], or when the number of dependent variables equals to one [1, 12, 13]. Through numerous counterexamples, we know that one cannot expect much regularity for weak local minimizers in general [2, 4, 9], and even strong local minimizers in high dimensional cases are only partially regular [3, 8, 9, 14, 15, 16].

Our main result is a link between uniform minimality property and H^2_{loc} regularity. So far there are two cases where uniform minimality property is known to hold. One is the case of C^1 weak local minimizers that satisfy uniform positivity of second variation. The other case is that of Lipschitz extremals and uniformly convex Lagrangian. In the first case our result can be obtained by standard regularity arguments. However, the argument in this paper is much simpler and more transparent. In the second case, our result is new and sheds more light on the regularity issue. The result holds in any number of space dimensions. A counterexample of Šverák and Yan [15] shows that if the number of independent variables is at least 3 and the number of dependent variables is at least 5 then our everywhere regularity cannot be improved by much and the partial regularity results of Evans [3] and Kristensen and Taheri [9] are the best one can expect in general.

In prior work people made various convexity assumptions on the Lagrangian and proved regularity or partial regularity of extremals or local or global minimizers. In this paper, the assumptions are placed on the behavior of the variational functional in the vicinity of the critical point rather than on the Lagrangian. The uniform minimality property can be easily shown to imply uniform quasiconvexity at the extremal (not uniform quasiconvexity everywhere, customarily assumed in regularity papers) and uniform positivity of second variation.

In the case of C^1 extremals the reverse implication holds as well [5, 6]. In the general case of Lipschitz extremals, our assumptions are strictly stronger than uniform quasiconvexity and uniform positivity of second variation, as implied by Corollary 7.3 in [9]. It is an important open problem to understand what other conditions one needs to place on the Lagrangian to ensure the uniform minimality property.

The idea of the proof comes from the well-known observation that inner variations lead to the Noether equation in the same way outer variations lead to the Euler-Lagrange equation. If the extremal is Lipschitz and H^2_{loc} then the Euler-Lagrange equation implies Noether equation. Our idea, studied more systematically in [7], is that inner variations could be understood as motions of singularities. Thus, singularities of Šverák-Yan, where the extremal is of class H^2 are not detectable by variational means.

In this paper we use the following system of notation. $|\boldsymbol{A}|$ denotes the Euclidean norm, if $\boldsymbol{A}$ is a vector and Frobenius norm $\sqrt{\mathrm{Tr}\,(\boldsymbol{A}\boldsymbol{A}^t)}$ if $\boldsymbol{A}$ is a matrix. $\|\boldsymbol{f}\|_p$ denotes the L^p norm of $|\boldsymbol{f}(\boldsymbol{x})|$. We use the inner product notation $(\boldsymbol{A},\boldsymbol{B})$ for the dot product if $\boldsymbol{A}$ and $\boldsymbol{B}$ are vectors and for the standard inner product $(\boldsymbol{A},\boldsymbol{B})=\mathrm{Tr}\,(\boldsymbol{A}\boldsymbol{B}^t)$, if $\boldsymbol{A}$ and $\boldsymbol{B}$ are matrices of the same shape. We also use indexless subscript notation for derivatives, such as $W_{\boldsymbol{F}}$ or $W_{\boldsymbol{FF}}$ for $\dfrac{\partial W}{\partial F_{ij}}$ and $\dfrac{\partial^2 W}{\partial F_{ij}\partial F_{kl}}$, respectively. Following Evans [3] we define $C^1_0(\Omega)\overset{\text{def}}{=}\{\phi\in C^1(\overline{\Omega})\colon \phi=0\text{ on }\partial\Omega\}$, which is *not* the closure of $C^\infty_0(\Omega)$ in C^1 topology.

2 Preliminaries

Let Ω be an open and bounded domain in $\mathbb{R}^d$. Let the Lagrangian $W\colon\overline{\Omega}\times\mathbb{R}^m\times\mathbb{M}\to\mathbb{R}$ be a continuous function, where $\mathbb{M}=\mathbb{R}^{m\times d}$. Consider the functional

$$E(\boldsymbol{y})=\int_\Omega W(\boldsymbol{x},\boldsymbol{y}(\boldsymbol{x}),\boldsymbol{F}(\boldsymbol{x}))d\boldsymbol{x},\quad \boldsymbol{F}(\boldsymbol{x})=\nabla\boldsymbol{y}(\boldsymbol{x}) \tag{2.1}$$

defined on the set of admissible functions

$$\mathcal{A}=\left\{\boldsymbol{y}\in W^{1,\infty}(\Omega;\mathbb{R}^m):\boldsymbol{y}(\boldsymbol{x})=\boldsymbol{g}(\boldsymbol{x}),\ \boldsymbol{x}\in\partial\Omega\right\}, \tag{2.2}$$

where $\boldsymbol{g}\in W^{1,\infty}(\partial\Omega;\mathbb{R}^m)$. Let $\mathrm{Var}(\mathcal{A})=W^{1,\infty}_0(\Omega;\mathbb{R}^m)$. Then $\boldsymbol{y}+\boldsymbol{\phi}\in\mathcal{A}$, for all $\boldsymbol{y}\in\mathcal{A}$ and all $\boldsymbol{\phi}\in\mathrm{Var}(\mathcal{A})$. Let

$$\Delta E(\boldsymbol{\phi})=\int_\Omega\{W(\boldsymbol{x},\boldsymbol{y}(\boldsymbol{x})+\boldsymbol{\phi},\boldsymbol{F}(\boldsymbol{x})+\nabla\boldsymbol{\phi})-W(\boldsymbol{x},\boldsymbol{y}(\boldsymbol{x}),\boldsymbol{F}(\boldsymbol{x}))\}d\boldsymbol{x}, \tag{2.3}$$

where

$$\boldsymbol{F}(\boldsymbol{x})=\nabla\boldsymbol{y}(\boldsymbol{x}).$$

Definition 2.1 *We say that $\boldsymbol{y}\in\mathcal{A}$ is a uniform $W^{1,\infty}$ (or weak) local minimizer, if there is $\beta>0$, such that for every sequence $\{\boldsymbol{\phi}_n\colon n\ge 1\}\subset\mathrm{Var}(\mathcal{A})$ such that $\boldsymbol{\phi}_n\to\boldsymbol{0}$ in $W^{1,\infty}(\Omega;\mathbb{R}^m)$*

$$\varliminf_{n\to\infty}\frac{\Delta E(\boldsymbol{\phi}_n)}{\|\nabla\boldsymbol{\phi}_n\|_2^2}\ge\beta. \tag{2.4}$$

Definition 2.2 *We say that* $\boldsymbol{y} \in \mathcal{A}$ *is a uniform* $W^{1,\infty}$ *weak-* local minimizer, if there is* $\beta > 0$*, such that for every sequence* $\{\boldsymbol{\phi}_n \colon n \geq 1\} \subset \mathrm{Var}(\mathcal{A})$ *such that* $\boldsymbol{\phi}_n \overset{*}{\rightharpoonup} \boldsymbol{0}$ *in* $W^{1,\infty}(\Omega; \mathbb{R}^m)$ *the inequality (2.4) holds.*

If we take $\boldsymbol{\phi}_n(\boldsymbol{x}) = \epsilon_n \boldsymbol{\phi}(\boldsymbol{x})$, where $\epsilon_n \to 0$, as $n \to \infty$, we easily see that the uniform minimizer must satisfy the uniform positivity of second variation condition $\delta^2 E(\boldsymbol{\phi}) \geq \|\nabla \boldsymbol{\phi}_n\|_2^2$, where

$$\delta^2 E(\boldsymbol{\phi}) = \int_\Omega \{(W_{\boldsymbol{yy}}(\boldsymbol{x})\boldsymbol{\phi}, \boldsymbol{\phi}) + 2(W_{\boldsymbol{Fy}}(\boldsymbol{x})\boldsymbol{\phi}, \nabla\boldsymbol{\phi}) + (W_{\boldsymbol{FF}}(\boldsymbol{x})\nabla\boldsymbol{\phi}, \nabla\boldsymbol{\phi})\} d\boldsymbol{x}, \tag{2.5}$$

where the second derivatives of W are evaluated at $(\boldsymbol{x}, \boldsymbol{y}(\boldsymbol{x}), \boldsymbol{F}(\boldsymbol{x}))$. If we take $\boldsymbol{\phi}_n(\boldsymbol{x}) = \epsilon_n \boldsymbol{\phi}((\boldsymbol{x} - \boldsymbol{a})/\epsilon_n)$, where $\boldsymbol{a} \in \Omega$, then we get a uniform quasiconvexity condition at $\boldsymbol{F}(\boldsymbol{a})$, provided $\boldsymbol{a}$ is a Lebesgue point of $\boldsymbol{F}(\boldsymbol{x}) = \nabla \boldsymbol{y}(\boldsymbol{x})$. For general $\boldsymbol{a}$ we get a new kind of quasiconvexity condition. This is because the sequence $\boldsymbol{F}(\boldsymbol{a} + \epsilon \boldsymbol{z})$ may not even converge almost everywhere. In that case, the new quasiconvexity condition would involve the Young measure $\nu_{\boldsymbol{z}}^{\boldsymbol{a}}$ of $\boldsymbol{F}(\boldsymbol{a} + \epsilon \boldsymbol{z})$:

$$\int_\Omega \int_{\mathbb{M}} \{W(\boldsymbol{F} + \nabla\boldsymbol{\phi}(\boldsymbol{z})) - W(\boldsymbol{F})\} d\nu_{\boldsymbol{z}}^{\boldsymbol{a}}(\boldsymbol{F}) d\boldsymbol{z} \geq \|\nabla\boldsymbol{\phi}\|_2^2. \tag{2.6}$$

However, the new condition (2.6) should not be taken as the appropriate generalization of quasiconvexity condition for Lipschitz extremals. If $\boldsymbol{F}(\boldsymbol{a} + \epsilon_n \boldsymbol{z})$ converges only weakly and $\nabla \boldsymbol{\phi}_n$ converges weakly as well, then the inequality (2.4) cannot be expressed as (2.6). The exact formulation of (2.4) in terms of the Lagrangian is at present unknown.

3 Higher regularity

For a Lipschitz function $\boldsymbol{y}(\boldsymbol{x})$ we define the compact set $\mathcal{R}(\boldsymbol{y})$ as follows.

$$\mathcal{R}_0(\boldsymbol{y}) = \{(\boldsymbol{y}(\boldsymbol{x}), \nabla \boldsymbol{y}(\boldsymbol{x})) : \boldsymbol{x} \text{ is a Lebesgue point of } \nabla \boldsymbol{y}\}, \quad \mathcal{R}(\boldsymbol{y}) = \overline{\mathcal{R}_0(\boldsymbol{y})}.$$

Let $\mathcal{O}$ be an open neighborhood of $\mathcal{R}(\boldsymbol{y})$ in $\mathbb{R}^m \times \mathbb{M}$.

THEOREM **3.1** *Assume that* $W \in C^2(\overline{\Omega} \times \mathcal{O})$*. Suppose that either* $\boldsymbol{y} \in C^1(\overline{\Omega}; \mathbb{R}^m)$ *and is a uniform weak local minimizer or* $\boldsymbol{y} \in \mathcal{A}$ *is a uniform* $W^{1,\infty}$ *weak-* local minimizer in the sense of Definitions 2.1 and 2.2. Then* $\boldsymbol{y} \in H^2_{loc}(\Omega; \mathbb{R}^m)$*. More precisely,* $h(\boldsymbol{x})\nabla \boldsymbol{y}(\boldsymbol{x}) \in H^1_0(\Omega; \mathbb{M})$ *for all* $h \in H^1_0(\Omega)$*.*

PROOF: Consider the inner variation

$$\boldsymbol{x} \mapsto \boldsymbol{x} + \epsilon \boldsymbol{h}(\boldsymbol{x}), \tag{3.1}$$

where $\boldsymbol{h} \in C^1_0(\Omega; \mathbb{R}^d)$. When $\epsilon < \|\nabla \boldsymbol{h}\|_\infty^{-1}$ the map (3.1) is a diffeomorphism of Ω onto itself.

Let us examine the increment of the functional $E(\boldsymbol{y})$ when $\boldsymbol{y}(\boldsymbol{x})$ is replaced by a "competitor" $\boldsymbol{y}_\epsilon(\boldsymbol{x}) = \boldsymbol{y}(\boldsymbol{x}_\epsilon(\boldsymbol{x}))$, where $\boldsymbol{x}_\epsilon(\boldsymbol{x})$ is the inverse of the diffeomorphism $\boldsymbol{x} \mapsto \boldsymbol{x} + \epsilon \boldsymbol{h}(\boldsymbol{x})$.

Observe that $\boldsymbol{y}_\epsilon(\boldsymbol{x}) \to \boldsymbol{y}(\boldsymbol{x})$ in $C^1(\overline{\Omega};\mathbb{R}^m)$, as $\epsilon \to 0$, if $\boldsymbol{y} \in C^1(\overline{\Omega};\mathbb{R}^m)$. In general, when $\boldsymbol{y}(\boldsymbol{x})$ is merely Lipschitz continuous, $\boldsymbol{y}_\epsilon(\boldsymbol{x}) \overset{*}{\rightharpoonup} \boldsymbol{y}(\boldsymbol{x})$ in $W^{1,\infty}(\Omega;\mathbb{R}^m)$ weak-*, as $\epsilon \to 0$. Thus, in either case, the increment of the functional

$$\Delta E(\epsilon) = \int_\Omega W(\boldsymbol{x}, \boldsymbol{y}(\boldsymbol{x}_\epsilon(\boldsymbol{x})), \boldsymbol{F}(\boldsymbol{x}_\epsilon(\boldsymbol{x}))\nabla \boldsymbol{x}_\epsilon(\boldsymbol{x}))d\boldsymbol{x} - \int_\Omega W(\boldsymbol{x}, \boldsymbol{y}(\boldsymbol{x}), \boldsymbol{F}(\boldsymbol{x}))d\boldsymbol{x}, \tag{3.2}$$

has a local minimum at $\epsilon = 0$. Therefore,

$$\left.\frac{d(\Delta E(\epsilon))}{d\epsilon}\right|_{\epsilon=0} = 0.$$

Notice that we can not differentiate under the integral sign in (3.2), because $\boldsymbol{F}(\boldsymbol{x})$ is not assumed to be smooth. However, differentiation under the integral sign will be permitted if we make a change of variables $\boldsymbol{x}' = \boldsymbol{x}_\epsilon(\boldsymbol{x})$ in the first integral in (3.2). We obtain

$$\Delta E(\epsilon) = \int_\Omega V(\boldsymbol{x}, \epsilon \boldsymbol{h}(\boldsymbol{x}), \epsilon \nabla \boldsymbol{h}(\boldsymbol{x}))d\boldsymbol{x}, \tag{3.3}$$

where

$$V(\boldsymbol{x}, \boldsymbol{\xi}, \boldsymbol{G}) = W(\boldsymbol{x} + \boldsymbol{\xi}, \boldsymbol{y}(\boldsymbol{x}), \boldsymbol{F}(\boldsymbol{x})(\boldsymbol{I} + \boldsymbol{G})^{-1}) \det(\boldsymbol{I} + \boldsymbol{G}) - W(\boldsymbol{x}, \boldsymbol{y}(\boldsymbol{x}), \boldsymbol{F}(\boldsymbol{x})).$$

Now we can differentiate under the integral sign in (3.3) and obtain

$$0 = \left.\frac{d(\Delta E(\epsilon))}{d\epsilon}\right|_{\epsilon=0} = \int_\Omega \{(V_{\boldsymbol{\xi}}(\boldsymbol{x}, \boldsymbol{0}, \boldsymbol{0}), \boldsymbol{h}(\boldsymbol{x})) + (V_{\boldsymbol{G}}(\boldsymbol{x}, \boldsymbol{0}, \boldsymbol{0}), \nabla \boldsymbol{h}(\boldsymbol{x}))\}d\boldsymbol{x}. \tag{3.4}$$

Equation (3.4) is equivalent to the Noether equation

$$-\nabla \cdot \boldsymbol{P}^* + W_{\boldsymbol{x}} = \boldsymbol{0}, \tag{3.5}$$

where

$$\boldsymbol{P}^*(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{F}) = W(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{F})\boldsymbol{I} - \boldsymbol{F}^t W_{\boldsymbol{F}}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{F})$$

is variously called the Eshelby, energy-momentum or Hamilton tensor.

Notice, that due to (3.4)

$$\Delta E(\epsilon) = \int_\Omega \{V(\boldsymbol{x}, \epsilon \boldsymbol{h}, \epsilon \nabla \boldsymbol{h}) - \epsilon(V_{\boldsymbol{\xi}}(\boldsymbol{x}, \boldsymbol{0}, \boldsymbol{0}), \boldsymbol{h}) - \epsilon(V_{\boldsymbol{G}}(\boldsymbol{x}, \boldsymbol{0}, \boldsymbol{0}), \nabla \boldsymbol{h})\}d\boldsymbol{x}.$$

By the Taylor expansion theorem, there exists a constant $K > 0$, depending on W and $\|\boldsymbol{F}\|_\infty$, such that for all $\boldsymbol{h} \in C_0^1(\Omega;\mathbb{R}^d)$ and all $0 < \epsilon < \|\boldsymbol{h}\|_{1,\infty}^{-1}$ we have

$$|V(\boldsymbol{x}, \epsilon \boldsymbol{h}, \epsilon \nabla \boldsymbol{h}) - \epsilon(V_{\boldsymbol{\xi}}(\boldsymbol{x}, \boldsymbol{0}, \boldsymbol{0}), \boldsymbol{h}) - \epsilon(V_{\boldsymbol{G}}(\boldsymbol{x}, \boldsymbol{0}, \boldsymbol{0}), \nabla \boldsymbol{h})| \le K\epsilon^2\{|\boldsymbol{h}|^2 + |\nabla \boldsymbol{h}|^2\}.$$

By the Poincaré inequality there exists a constant $C > 0$, depending on W, $\|\boldsymbol{F}\|_\infty$ and Ω, such that

$$\overline{\lim_{\epsilon\to 0}} \frac{|\Delta E(\epsilon)|}{\|\epsilon\nabla\boldsymbol{h}\|_2^2} \le C \tag{3.6}$$

for all $\boldsymbol{h} \in C_0^1(\Omega;\mathbb{R}^d)$.

The upper bound (3.6) is supplemented with the lower bound coming from (2.4). To apply (2.4) we define

$$\boldsymbol{\phi}_\epsilon(\boldsymbol{x}) = \boldsymbol{y}(\boldsymbol{x}_\epsilon(\boldsymbol{x})) - \boldsymbol{y}(\boldsymbol{x}). \tag{3.7}$$

Observe that $\Delta E(\epsilon) = \Delta E(\boldsymbol{\phi}_\epsilon)$, where the left-hand side is defined in (3.2) and the right-hand side is defined in (2.3). Applying (3.6) and (2.4), we obtain

$$\overline{\lim_{\epsilon\to 0}} \frac{\|\nabla\boldsymbol{\phi}_\epsilon\|_2^2}{\|\epsilon\nabla\boldsymbol{h}\|_2^2} = \overline{\lim_{\epsilon\to 0}} \frac{\dfrac{|\Delta E(\epsilon)|}{\|\epsilon\nabla\boldsymbol{h}\|_2^2}}{\dfrac{|\Delta E(\boldsymbol{\phi}_\epsilon)|}{\|\nabla\boldsymbol{\phi}_\epsilon\|_2^2}} \le \frac{C}{\beta} \tag{3.8}$$

for all $\boldsymbol{h} \in C_0^1(\Omega;\mathbb{R}^d)$. The estimate (3.8) is the direct consequence of the uniform local minimality property of $\boldsymbol{y}(\boldsymbol{x})$. Our next lemma makes it clear why the inequality (3.8) is related to higher regularity of $\boldsymbol{y}(\boldsymbol{x})$.

Lemma **3.2** *There exists a constant $C > 0$, depending only on the bound in (3.8) and $\|\boldsymbol{F}\|_\infty$, so that for all $\boldsymbol{h} \in C_0^1(\Omega;\mathbb{R}^d)$*

$$\overline{\lim_{\epsilon\to 0}} \frac{1}{\|\epsilon\nabla\boldsymbol{h}\|_2^2} \int_\Omega |\boldsymbol{F}(\boldsymbol{x}) - \boldsymbol{F}(\boldsymbol{x} + \epsilon\boldsymbol{h}(\boldsymbol{x}))|^2 d\boldsymbol{x} \le C. \tag{3.9}$$

Proof: Changing variables $\boldsymbol{x}' = \boldsymbol{x}_\epsilon(\boldsymbol{x})$ in the numerator in the left-hand side in (3.8), we obtain

$$\overline{\lim_{\epsilon\to 0}} \frac{\|\nabla\boldsymbol{\phi}_\epsilon\|_2^2}{\|\epsilon\nabla\boldsymbol{h}\|_2^2} = \overline{\lim_{\epsilon\to 0}} \frac{1}{\|\epsilon\nabla\boldsymbol{h}\|_2^2} \int_\Omega |\boldsymbol{F}(\boldsymbol{x}')(\boldsymbol{I} + \epsilon\nabla\boldsymbol{h})^{-1} - \boldsymbol{F}(\boldsymbol{x}' + \epsilon\boldsymbol{h}(\boldsymbol{x}'))|^2 d\boldsymbol{x}', \tag{3.10}$$

where we have discarded the term $\det(\boldsymbol{I} + \epsilon\nabla\boldsymbol{h})$, since it converges to 1, uniformly as $\epsilon \to 0$.

Next, we eliminate $(\boldsymbol{I} + \epsilon\nabla\boldsymbol{h})^{-1}$ from (3.10), using the fact that $\boldsymbol{I} + \epsilon\nabla\boldsymbol{h}$ is uniformly close to $\boldsymbol{I}$, when ϵ is small. We get,

$$\overline{\lim_{\epsilon\to 0}} \frac{\|\nabla\boldsymbol{\phi}_\epsilon\|_2^2}{\|\epsilon\nabla\boldsymbol{h}\|_2^2} = \overline{\lim_{\epsilon\to 0}} \frac{1}{\|\epsilon\nabla\boldsymbol{h}\|_2^2} \int_\Omega |\boldsymbol{F}(\boldsymbol{x}) - \boldsymbol{F}(\boldsymbol{x} + \epsilon\boldsymbol{h}(\boldsymbol{x}))(\boldsymbol{I} + \epsilon\nabla\boldsymbol{h}(\boldsymbol{x}))|^2 d\boldsymbol{x}. \tag{3.11}$$

Equation (3.11) follows from (3.10) and a simple inequality from the theory of matrices.

Lemma **3.3** *Let σ_{min} and σ_{max} be the minimal and maximal singular values, respectively, of a $d \times d$ matrix $\boldsymbol{A}$. Then*

$$\sigma_{min}|\boldsymbol{B}| \le |\boldsymbol{B}\boldsymbol{A}| \le \sigma_{max}|\boldsymbol{B}|$$

for all $m \times d$ matrices $\boldsymbol{B}$.

Proof: $|\boldsymbol{B}\boldsymbol{A}|^2 = \mathrm{Tr}\,(\boldsymbol{A}\boldsymbol{A}^t\boldsymbol{B}^t\boldsymbol{B})$. Observe that $\boldsymbol{A}\boldsymbol{A}^t \geq \sigma_{\min}^2\boldsymbol{I}$ and

$$|\boldsymbol{B}\boldsymbol{A}|^2 = \mathrm{Tr}\,((\boldsymbol{A}\boldsymbol{A}^t - \sigma_{\min}^2\boldsymbol{I})\boldsymbol{B}^t\boldsymbol{B}) + \sigma_{\min}^2|\boldsymbol{B}|^2. \tag{3.12}$$

By a theorem of Schur (see e.g. [10, Theorem 10.7]), the first term on the right hand side of (3.12) is non-negative, since the matrices $\boldsymbol{A}\boldsymbol{A}^t - \sigma_{\min}^2\boldsymbol{I}$ and $\boldsymbol{B}^t\boldsymbol{B}$ are symmetric and non-negative definite. Similarly,

$$|\boldsymbol{B}\boldsymbol{A}|^2 = \sigma_{\max}^2|\boldsymbol{B}|^2 - \mathrm{Tr}\,((\sigma_{\max}^2\boldsymbol{I} - \boldsymbol{A}\boldsymbol{A}^t)\boldsymbol{B}^t\boldsymbol{B}) \leq \sigma_{\max}^2|\boldsymbol{B}|^2.$$

▮

The inequality $|\boldsymbol{a}+\boldsymbol{b}|^2 \leq 2|\boldsymbol{a}|^2 + 2|\boldsymbol{b}|^2$, written as

$$|\boldsymbol{F}(\boldsymbol{x}) - \boldsymbol{F}(\boldsymbol{x}+\epsilon\boldsymbol{h})|^2 \leq 2|\boldsymbol{F}(\boldsymbol{x}) - \boldsymbol{F}(\boldsymbol{x}+\epsilon\boldsymbol{h}) - \boldsymbol{F}(\boldsymbol{x}+\epsilon\boldsymbol{h})\epsilon\nabla\boldsymbol{h}|^2 + 2|\boldsymbol{F}(\boldsymbol{x}+\epsilon\boldsymbol{h})\epsilon\nabla\boldsymbol{h}|^2,$$

together with (3.11) implies

$$\overline{\lim_{\epsilon\to 0}}\frac{1}{\|\epsilon\nabla\boldsymbol{h}\|_2^2}\int_\Omega |\boldsymbol{F}(\boldsymbol{x}) - \boldsymbol{F}(\boldsymbol{x}+\epsilon\boldsymbol{h}(\boldsymbol{x}))|^2 d\boldsymbol{x} \leq 2\,\overline{\lim_{\epsilon\to 0}}\frac{\|\nabla\boldsymbol{\phi}_\epsilon\|_2^2}{\|\epsilon\nabla\boldsymbol{h}\|_2^2} + 2\|\boldsymbol{F}\|_\infty^2.$$

Lemma 3.2 now follows from (3.8). ▮

The following Lemma applied to every component of the matrix field $\boldsymbol{F}(\boldsymbol{x})$ will finish the proof of Theorem 3.1.

Lemma **3.4** *Let Ω be an open bounded domain in $\mathbb{R}^d$. Let $f \in L^\infty(\Omega)$ be such that for all $\boldsymbol{h} \in C_0^1(\Omega;\mathbb{R}^d)$*

$$\overline{\lim_{\epsilon\to 0}}\frac{1}{\epsilon^2}\int_\Omega |f(\boldsymbol{x}+\epsilon\boldsymbol{h}(\boldsymbol{x})) - f(\boldsymbol{x})|^2 d\boldsymbol{x} \leq C\int_\Omega |\nabla\boldsymbol{h}(\boldsymbol{x})|^2 d\boldsymbol{x}. \tag{3.13}$$

Then $f \in H^1_{loc}(\Omega)$ and $h\nabla f \in L^2(\Omega)$ for all $h \in H_0^1(\Omega)$.

Proof: In view of (3.13) there exists a subsequence, not relabeled and a function g (both dependent on $\boldsymbol{h}$) such that

$$\frac{f(\boldsymbol{x}+\epsilon\boldsymbol{h}(\boldsymbol{x})) - f(\boldsymbol{x})}{\epsilon} \rightharpoonup g$$

weakly in $L^2(\Omega)$. In particular

$$\lim_{\epsilon\to 0}\frac{1}{\epsilon}\left[\int_\Omega f(\boldsymbol{x}+\epsilon\boldsymbol{h}(\boldsymbol{x}))\phi(\boldsymbol{x})d\boldsymbol{x} - \int_\Omega f(\boldsymbol{x})\phi(\boldsymbol{x})d\boldsymbol{x}\right] = \int_\Omega g(\boldsymbol{x})\phi(\boldsymbol{x})d\boldsymbol{x}, \tag{3.14}$$

for all $\phi \in C_0^\infty(\Omega)$. Making change of variables $\boldsymbol{x}' = \boldsymbol{x}+\epsilon\boldsymbol{h}(\boldsymbol{x})$ in the first integral in (3.14), we get

$$\int_\Omega g\phi d\boldsymbol{x} = \lim_{\epsilon\to 0}\int_\Omega f(\boldsymbol{x}')\left(\frac{\phi(\boldsymbol{x}_\epsilon(\boldsymbol{x}'))\det(\nabla\boldsymbol{x}_\epsilon(\boldsymbol{x}')) - \phi(\boldsymbol{x}')}{\epsilon}\right)d\boldsymbol{x}'.$$

Using the fact that $\boldsymbol{x}_\epsilon(\boldsymbol{x}) \to \boldsymbol{x}$ in $C^1(\overline{\Omega};\mathbb{R}^d)$, we obtain

$$\int_\Omega g\phi d\boldsymbol{x} = -\int_\Omega f\nabla\cdot(\phi\boldsymbol{h})d\boldsymbol{x}.$$

It follows that $\nabla\cdot(f\boldsymbol{h}) = g + f\nabla\cdot\boldsymbol{h} \in L^2(\Omega)$ in the sense of distributions.

Now let $\boldsymbol{h}(\boldsymbol{x}) = h(\boldsymbol{x})\boldsymbol{e}_i$ for some $h \in C_0^1(\Omega)$, where $\boldsymbol{e}_i$ is the i^{th} standard basis vector. Then

$$\frac{\partial}{\partial x_i}(f(\boldsymbol{x})h(\boldsymbol{x})) = \nabla\cdot(f(\boldsymbol{x})\boldsymbol{h}(\boldsymbol{x})) \in L^2(\Omega)$$

This implies that $f \in H^1_{loc}(\Omega)$. Thus, it follows from (3.13) that

$$\int_\Omega (\nabla f, \boldsymbol{h})^2 d\boldsymbol{x} \le C\|\nabla\boldsymbol{h}\|_2^2 \tag{3.15}$$

for all $\boldsymbol{h} \in C_0^\infty(\Omega;\mathbb{R}^d)$.

In order to prove the last claim in the Lemma, we fix $\boldsymbol{h} \in H_0^1(\Omega;\mathbb{R}^d)$ and consider a sequence $\{\boldsymbol{h}_n : n \ge 1\} \subset C_0^\infty(\Omega;\mathbb{R}^d)$, such that $\boldsymbol{h}_n \to \boldsymbol{h}$ in $H_0^1(\Omega;\mathbb{R}^d)$. It follows that there is a subsequence (not relabeled) such that $\boldsymbol{h}_n \to \boldsymbol{h}$ for almost every $\boldsymbol{x} \in \Omega$. By Fatou's lemma

$$\int_\Omega (\nabla f(\boldsymbol{x}), \boldsymbol{h}(\boldsymbol{x}))^2 d\boldsymbol{x} \le \varliminf_{n\to\infty} \int_\Omega (\nabla f(\boldsymbol{x}), \boldsymbol{h}_n(\boldsymbol{x}))^2 d\boldsymbol{x} \le C\|\nabla\boldsymbol{h}\|_2^2.$$

Taking $\boldsymbol{h}(\boldsymbol{x}) = h(\boldsymbol{x})\boldsymbol{e}_i$ finishes the proof of the Lemma. ▮

Theorem 3.1 is now proved. ▮

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