Computer-Generated Symmetric Chain Decompositions for L(4,n) and L(3,n)

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1. Introduction

Recall that the famous *Young's partition lattice* $L(m,n)$ consists of the set of integer-vectors

$$(a_1, a_2, \cdots, a_m), \quad 0 \le a_1 \le a_2 \le \cdots \le a_m \le n,$$

with the order relation

$$\vec{a} \le \vec{b} \quad \text{if} \quad a_i \le b_i \quad \text{for} \quad i = 1, 2, \cdots, m.$$

The **rank** r is defined by

$$r(\vec{a}) = \sum_{i=1}^{m} a_i.$$

And recall that a chain $\vec{v_1} \le \vec{v_2} \le \cdots \le \vec{v_k}$ in $L(m,n)$ is called **saturated** if it skips no ranks and **symmetric** if $r(\vec{v_1}) + r(\vec{v_k}) = mn$.

A **Symmetric chain decomposition**(SCD) of a poset is a way of expressing it as a disjoint union of saturated symmetric chains.

One of major problems in order theory is the explicit construction of SCDs for Young's Lattice for *all* m and n. In 1989, Kathy O'Hara ([2], see also [5]) has astounded the combinatorial world by constructing SCDs for the 'trivial extension' of $L(m,n)$, in which all partitions of one rank are related to the next; but the problem is remaining wide open for Young's lattice itself.

SCDs for L(4,n) and L(3,n) have been constructed by West[4] and Lindström[1]. In this paper we explicitly provide **complete** SCDs for L(4,n) and L(3,n), which were found by the assistance of our computer. And far more interestingly, it is proved *completely automatically* without using any human help (except for writing the general Maple program). While our construction for $L(4,n)$ is not equivalent to West's construction, it is nevertheless of the similar format. On the other hand, our construction for $L(3,n)$ is more elegant than Lindström's, since it is not split into even and odd cases.

We hope the present approach would ultimately lead to computer-generated or at least computer-assisted constructions of SCDs for $L(m,n)$, or at least for $L(5,n)$. Meanwhile we are unable to do the case of $L(5,n)$. We also hope the present *methodology* will be useful for future attacks on this challenging and tantalizing problem.

2. New SCDs for L(3,n) and L(4,n)

Theorem 1 *The following tables give symmetric chain decompositions for* $L(3,n)$ *and* $L(4,n)$*. In tables below,* i, j *and* k *are generic non-negative integers, and vertical dots represent that the only component that is not the same gets decremented by* 1*. For example,* $(n-i-3j, n-i-2j, n-j) \ldots (i+j, n-i-2j, n-j)$ *is a shorthand for the chain:*

$$(n-i-3j-a, n-i-2j, n-j), a = 0 \ldots n-2i-4j.$$

C_{ij} $2i+4j \le n$			D_{ij} $2i+4j+3 \le n$		
(n-i-3j,	*n-i-2j,*	*n-j)*	*(n-i-3j-1,*	*n-i-2j-1,*	*n-j-1)*
⋮					⋮
(i+j,	*n-i-2j,*	*n-j)*	*(n-i-3j-1,*	*n-i-2j-1,*	*n-i-j-1)*
	⋮		⋮		
(i+j,	*i+2j,*	*n-j)*	*(j,*	*n-i-2j-1,*	*n-i-j-1)*
		⋮		⋮	
(i+j,	*i+2j,*	*i+3j)*	*(j,*	*i+2j+1,*	*n-i-j-1)*
⋮					⋮
(j,	*i+2j,*	*i+3j)*	*(j,*	*i+2j+1,*	*i+3j+2)*

A complete SCD for L(3,n)

C_{ijk} $2i+2j+3k \le n$				D_{ijk} $2i+2j+3k+3 \le n$			
(n-2k-2j-i,	*n-2k-j-i,*	*n-k-j,*	*n-k)*	*(n-2k-2j-i-1,*	*n-2k-j-i-1,*	*n-k-j-1,*	*n-k)*
⋮						⋮	
(k+i,	*n-2k-j-i,*	*n-k-j,*	*n-k)*	*(n-2k-2j-i-1,*	*n-2k-j-i-1,*	*n-k-i-j-1,*	*n-k)*
	⋮						⋮
(k+i,	*k+j+i,*	*n-k-j,*	*n-k)*	*(n-2k-2j-i-1,*	*n-2k-j-i-1,*	*n-k-i-j-1,*	*n-k-i-1)*
		⋮		⋮			
(k+i,	*k+j+i,*	*2k+j+i,*	*n-k)*	*(k,*	*n-2k-j-i-1,*	*n-k-i-j-1,*	*n-k-i-1)*
			⋮		⋮		
(k+i,	*k+j+i,*	*2k+j+i,*	*2k+2j+i)*	*(k,*	*k+j,*	*n-k-i-j-1,*	*n-k-i-1)*
⋮						⋮	
(k,	*k+j+i,*	*2k+j+i,*	*2k+2j+i)*	*(k,*	*k+j,*	*2k+j+i+1,*	*n-k-i-1)*
	⋮						⋮
(k,	*k+j,*	*2k+j+i,*	*2k+2j+i)*	*(k,*	*k+j,*	*2k+j+i+1,*	*2k+2j+i+2)*

A complete SCD for L(4,n)

Proof: The chains are clearly saturated and symmetric. Thus we only need to prove that each vector in $L(m, n)$ $(m = 3, 4)$ appears only once in the tables. We introduce the commuting indeterminate $x_1, x_2, \cdots, x_m$ and t, and define the **weight** for a vector $\vec{a} = (a_1, a_2, \cdots, a_m)$ in $L(m, n)$ as the following:

$$\mathrm{w}(\vec{a}) = (x_0 t)^{n-a_m}(x_1 t)^{a_m - a_{m-1}} \cdots (x_{m-1}t)^{a_2-a_1}(x_m t)^{a_1}$$

For a fixed m, it is easy to see that the total weight,

$$\sum_{n=0}^{\infty} \mathrm{w}(\vec{a}), \quad \text{where} \quad \vec{a} \in L(m, n),$$

is a generating function

$$G(t; x_1, x_2, \cdots, x_m) = \frac{1}{(1-x_0 t)(1-x_1 t)\cdots(1-x_m t)}.$$

Each term in the expanded power series of $G(t; x_1, x_2, \cdots, x_m)$ has coefficient 1 and corresponds to a unique vector in $L(m, n)$. On the other hand, for each vector in $L(m, n)$, there is a unique corresponding term in the power series of $G(t; x_1, x_2, \cdots, x_m)$. Therefore, to prove that each vector in $L(m, n)$ $(m = 3, 4)$ appears only once in the SCDs is the same as to prove that the total weights of the vectors in the given chains are $G(t; x_1, x_2, x_3)$ and $G(t; x_1, x_2, x_3, x_4)$ respectively. The part of summing over all the weights of the vectors is done by computer. ■

This method of proof can be applied to any conjectured SCDs. The difficult part is to find such decompositions, here we need human-computer interactions by using a modified greedy algorithm. Once it is found, the verification part is purely automatic by using the Maple program `Lmn`. `Lmn` can also be used to give completely automatic proofs of the validity of Lindström's and West's constructions.

For general m,n, an explicit construction of SCDs of $L(m,n)$ is still an open problem.

3. Maple package

The summation of all the weights of the vectors in the given chains is automatically done by computer. The Maple package is available at http://www.math.temple.edu/~wen/lattice/. After downloading the file to the local disk, type
>read("Lmn");
in the maple workspace. There is detailed on-line help on how to use the procedures in the package `Lmn`. Procedures to compute the total weights of the vectors in SCDs given by Lindström[1] and by West[4] are also included in the package `Lmn`.

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