↑name↑ ↑ID#↑ November 25, 2002. Show all work.

Math 75 – Fall 2002 – Quiz #3 – Warren D. Smith

1. For what function $F(x)$, is it the case that both $\frac{d}{dx}F(x) = 5 + 7x^3$ and $F(0) = 1$?
   ANSWER: $F(x) = 5x + 7x^4/4 + 1$.

2. Suppose x and y are related by $x\ln(y) = 5y + \cos(xy)$. Write a formula, in terms of y and x, for $y'(x)$.
   ANSWER: Implicit differentiation [left side by product rule, 2nd term on right side by chain rule] to get $xy'/y + \ln y = 5y' - \sin(xy)(xy' + y)$; rewrite with coefficients of y' and not of y' grouped:
   $[x/y - 5 + \sin(xy)x]y' = -\sin(xy)y - \ln y$;
   then solve for y' to get answer $y' = (-\sin(xy)y - \ln y)/[x/y - 5 + \sin(xy)x]$.

3. What is the area of of the region of the xy plane represented by $0 < x < 2$ and $0 < y < 3\sin(x/5)$?
   (a) Sketch.
   (b) Write down this area as an integral.
   (c) Do the integral, i.e. write the answer as a formula.

   ANSWERS: (b) Area= $\int_0^2 3\sin(x/5)\,dx$.
   (c) If $F(x) = -15\cos(x/5)$ then $F'(x) = 3\sin(x/5)$. [One way to find this "magic" $F(x)$: postulate $F(x) = Q\cos(Rx)$ for some as-yet-unknown Q & R, then differentiate to get $F'(x) = -QR\sin(Rx)$, then by matching this up with $F'(x) = 3\sin(x/5)$ we see $R = 1/5$ and $QR = -3$ so that $Q = -15$.]
   (c) So Area= $F(2) - F(0) = 15\sin(2/5) - 15\sin(0/5) = 15\sin(2/5)$.

4. What is the derivative (d/dx) of the following functions of x:
   (a) $\arcsin(5x)\ln(x)$
   (b) $2^{\cos(x)}$
   (c) $\ln(x)^{99}$
   (d) $\arctan(x^2)$.
   ANSWERS: (a) $\frac{5\ln(x)}{\sqrt{1-(5x)^2}} + \frac{\arcsin(5x)}{x}$ [first use product rule, remembering $\arcsin'(x) = (1-x^2)^{-1/2}$ and $\ln'(x) = 1/x$ then use chain rule with inner$(x) = 5x$ and outer(z) =arcsin(z)]
   (b) $-(\ln 2)2^{\cos(x)}\sin(x)$ [first use $A^B = e^{B\ln A}$ to rewrite $2^{\cos(x)} = e^{\cos(x)\ln(2)}$ then use chain rule with inner$(x) = \cos(x)\ln(2)$ and outer$(z) = e^z$]
   ANOTHER WAY TO DO IT: Use "logarithmic differentiation" which is based on using the chain rule formula $\frac{d}{dx}\ln F(x) = F'(x)/F(x)$ in the "reverse direction:" $F'(x) = F(x)\frac{d}{dx}\ln F(x)$. Here we have $F(x) = 2^{\cos(x)}$ so $\ln F(x) = \cos(x)\ln 2$ by using the property $\ln(A^B) = B\ln A$ of the logarithm. So $\frac{d}{dx}\ln F(x) = -\sin(x)\ln 2$, so $F'(x) = -2^{\cos(x)}\sin(x)\ln 2$.
   (c) $99\ln(x)^{98}/x$ [Use $\frac{d}{dx}[F(x)^n] = nF(x)^{n-1}F'(x)$, here with $F(x) = \ln(x)$ and $f'(x) = 1/x$. This is turn (if you did not already know it) arises from the chain rule using outer function being raising to nth power, inner function being $F(x)$.]
   (d) $2x/(1 + x^4)$ [Use $\arctan'(x) = 1/(1 + x^2)$ and chain rule].

5. Find the Taylor series (only necessary to include terms up to and including x^2) of $F(x) = e^{\sin x}$ based at $x = 0$.
   Then as a sanity check, compare $F(1/5) \approx 1.2198$ with the exact value of your quadratic Taylor approximation at $1/5$.
   ANSWER: Start from Taylor series master formula $F(x) = F(0)+F'(0)x+F''(0)x^2/2+F'''(0)x^3/3!+\ldots$ and plug in these ingredients: $F(0) = e^0 = \mathbf{1}$;
   $F'(x) = e^{\sin(x)}\cos(x)$ [from chain rule], $F'(0) = e^0\cos(0) = \mathbf{1}$;
   $F''(x) = e^{\sin(x)}\cos(x)^2 - e^{\sin(x)}\sin(x)$ [from product rule], $F''(0) = e^0\cos(0)^2 - e^0\sin(0) = 1 - 0 = \mathbf{1}$;
   Therefore answer is: $F(x) = 1 + 1x + 1x^2/2 + \ldots$. That again, simplified: $F(x) = 1 + x + x^2/2 + \ldots$.

Sanity check: $F(1/5) \approx 1.2198$ is only 0.0002 away from $1 + (1/5) + (1/5)^2/2 = 1 + 0.2 + 0.02 = 1.22$. Note this quadratic approximation here has 100 times smaller error than the linear approximation!

6. Minimize $y = \sqrt{x - \ln(x) + 1}$ when $x > 0$. What is x and what is y at the minimum?

   ANSWER: It is easier to minimize y^2 than y. $(y^2) = x - \ln(x) + 1$. Take deriv: $(y^2)' = 1 - 1/x$. Set deriv$= 0$ to find critical point: $1 - 1/x = 0$ so ANSWER$_1$: $x = 1$. Then $y^2 = 1 - \ln(1) + 1 = 1 - 0 + 1 = 2$ so ANSWER$_2$: $y = \sqrt{2}$. SANITY CHECK: $y^2 = 2.00005$ when $x = 0.99$ (or when $x = 1.01$) and $y^2 = 2$ when $x = 1$. So this really is a min.

7. For which real x is $(x + 2)(x - 7)$ concave-∪?

   ANSWER: The derivative is $x + 2 + x - 7 = 2x - 5$. [By product rule; or: expand it out to get $x^2 - 5x - 14$ then find the derivative$= 2x - 5$.] The 2nd derivative is 2. We are concave-∪ when the 2nd derivative is *positive* i.e. when $2 > 0$, i.e. *always*.

8. What is the equation of the tangent line (linear approximation) to $Y = X^{1/4}$ based at $X = 16$? $Y_{\text{lin.app.}} = ?$

   ANSWER: $Y'(X) = X^{-3/4}/4$ so the slope of the tangent line is $Y'(16) = (1/8)/4 = 1/32$. And $Y(16) = 16^{1/4} = 2$. So the tangent line is

   $$Y_{\text{lin.app.}} = 2 + (X - 16)/32$$

   which has the correct slope, and is at the correct height when $X = 16$. [You could also have got this from the Taylor series formula $Y(X) = Y(B) + Y'(B)(x - B) + Y''(B)(X - B)^2/2! + \dots$ with basepoint B, taking only the first 2 terms on the right.]