$\uparrow$name$\uparrow$ $\uparrow$ID#$\uparrow$ April 15, 2003. (2 pages.)

Math 55 – Spr 2002 – Quiz 2 – Warren D. Smith. ANSWERS 180/180.

1. **There is always exactly ONE correct answer EXCEPT that IN JUST ONE OF THE QUESTIONS ON THIS TEST, there are TWO correct answers. Circling only 1 of those 2 means half credit. Circling a wrong answer means automatic zero credit. Calculators allowed and encouraged. Notes and books not allowed.**

   From among 13 animals, you pick 5. How many possible 5-animal sets could you have ended up with?
   ANS. a $= {}_{13}C_5 = 1287$. (Note: a "set" is the same even if you change the order you list its elements in. E.g., the set {A,B} is the same set as {B,A}. So use ${}_{13}C_5$ and NOT ${}_{13}P_5$.)

2. A line passes thru these two (x, y)-points: $(5, 7)$ and $(12, 56)$. What is the equation of the line?
   ANS. b: $y = 7x - 28$. The slope is rise/run=$(56 - 7)/(12 - 5) = 7$. Then can do sanity checks that b works: If $x = 5$ then get $y = 7 \times 5 - 28 = 7$, if $x = 12$ get $y = 7 \times 7 - 28 = 56$. Good.
   POPULAR **WRONG** ANS: e: $y = 2x - 3$. Fails when $x = 12$, would give $y = 2 \times 12 - 3 = 21$ instead of 56.

3. An exponentially growing population doubles in 27 years. In 15 years, by what factor (accurate to 3 or more decimals...) does it increase?
   ANS. c= 1.4697 and e= $2^{5/9}$ are both correct (and the same). Formula C says $2^{15/27}$, and note 15/27=5/9.

4. Prices increase 3% per year via exponential growth. How many years before prices double? (rounded off to 3 or more decimals...)
   ANS. b= 23.4498. Since $1.03^{23.4498} = 2$.
   ANOTHER WAY: Solve $1.03^x = 2$ for x, get $23.4498 = log(2)/log(1.03)$.
   WRONG ANS c= 33.3333 would have been correct in linear ("simple interest") growth, but this is *exponential* growth (multiply by 1.03 each year).

5. You pick 2 cards from a standard 52-card, 4-suit deck. What are the *odds* your two cards happen to be (in some order) consecutive of the same suit such as {3♣, 4♣} or {$Q\heartsuit$, $J\heartsuit$}? [Warning: {2♣, 3$\diamondsuit$} would *not* count as consecutive of same suit.]
   ANS (e) 8:213.
   Tough one. There are $12 \times 4 = 48$ ways the 2 cards could come out consecutive: Namely there are 48 cards the lower-numbered card could be (say, everything but an Ace in the usual 2,3,4,5,6,7,8,9,10,J,Q,K,A card-order) and then there is no choice about the higher-numbered card. There are ${}_{52}C_2 = 1326$ ways to pick 2 cards from a 52-card deck. So the ratio is: Prob $= 48/1326 = 8/221$. Finally, the *odds* are $8/221 : 213/221 =$ 8:213.

6. How many baseball lineups (9 people, in order – different orders count as different lineups) can you make from a 20-member squad?
   ANS c $= {}_{20}P_9 = 20 \times 19 \times 18 \times 17 \times 16 \times 15 \times 14 \times 13 \times 12 = 60949324800$
   Must use P not C since "different orders count as different." SANITY NOTE: since 9 terms here, each between 10 and 20, we expect a product bigger than 10^9, but... not bigger than 10^{11}. That alone is enough to narrow it down to choice c.

7. A license plate is made of a 3-letter "word" with no repeated letters, such as ABQ (but not AQA!) followed by a 3-digit number with each digit being from 0 to 9 (repeats allowed). [Examples of 4 license plates which all count as different: JBD-007, JAB-007, BDJ-007, JBD-070.] How many license plates of this form are possible?
   (a) $26^3 10^3$

(b) ${}_{26}C_3 10^3$
(c) ${}_{26}P_3 10^3$
(d) ${}_{26}C_3\ {}_{10}C_3$
(e) ${}_{26}C_3\ {}_{10}P_3$
ANS c= ${}_{26}P_3\ 10^3 = 15600000$.
Has to be a P since order matters. If you used b= ${}_{26}C_3 10^3$, then you'd wrongly be counting JBD-007 and BDJ-007 as the "same." If you chose a= $26^3 10^3$ then you'd wrongly be counting words with repeated letters such as AAA.

8. You toss a fair dice 7 times. (Possible rolls 1-6.). What is the probability you get a roll less than 3, at least once in these 7 tries?
ANS b=$1-(4/6)^7 = 2059/2187$. Use the "at least once" rule.
POPULAR WRONG ANS: $a = (4/6)^7 = 128/2187$. That would have been the probability of rolling ≥ 3 all 7 times – precisely NOT what was asked for. Correct is 1−that.

9. There are 5 men in a class of 7 students. If you line them up in random order, what is the probability that the first 5 in the line-of-7 will happen to all be men?
ANS. a= 5!2!/7!
There are 5! ways the men (first 5) could be ordered, 2! ways the women (last 2) could be ordered, versus 7! ways they all could be ordered. Prob = the ratio. ANOTHER WAY: This is $1/{}_7C_5$, since ${}_7C_5$ ways to choose 5 people from the class to be the first 5 in line, but only 1 of those ways is all men. YET ANOTHER: This is $\frac{5}{7}\frac{4}{6}\frac{3}{5}\frac{2}{4}\frac{1}{3}$.
WRONG ANS: d= $(5/7)^5(2/7)^2$ would have been right if all these probabilities had been independent, *but* they are dependent (if a man comes first in line, that affects the prob of who comes second in line). The last("yet another") right ans was doing it by correctly accounting for those dependencies..

10. There are 5 men, 7 women, 11 spiders, and 13 ants in a room. You pick one at random. Also you roll a fair dice. What is the probability that you get a {woman OR a spider}, AND, at the same time, your roll was a 3?
ANS d= 1/12. Being woman or spider is disjoint (can't be both!) so can use OR-add rule to get (7+11)/(5+7+11+13) = 1/2. And Prob(dice roll is 3)=1/6. These independent (neither event affects the chances of the other) so can use AND-multiply rule. So ans is $(1/2)\times(1/6) = 1/12$.
INSANE WRONG ANS: a="more than 2." Hello! Probabilities always are between 0 and 1. Get with it!

11. In the previous problem, what's the probability you pick {a man OR woman} AND you do NOT roll 5?
ANS e= 5/18. Being man or woman is disjoint so can use OR-add rule to get $(5+7)/36 = 1/3$ as prob of picking either. Prob you do NOT roll 5 is $1-1/6 = 5/6$. Now since independent (your gender does not affect the dice roll!) can use the AND-multiply rule to get final answer $(1/3)\times(5/6) = 5/18$.

12. You flip 15 fair coins. What is the expected number of heads?
ANS e= 7.5 = 15/2. Since expected number of heads per coin flip is 1/2.

13. You bring two calculators to a math test. The first has a probability of 0.05 of breaking. The second will break with (independent) probability 0.02. What is the probability that at least one of the calculators will work for you? [Hint: what way are these combined with AND, OR and NOTs?]
ANS. e= .9990. The probability that it is NOT the case that {calc#1 AND calc#2 both break} is $1-0.02\times0.05 = 1-0.001 = .9990$. Used the NOT rules and used the AND-multiply (independent events) rule.
POPULAR WRONG ANS. c= $0.9310 = (1-0.02)\times(1-0.05) = .98\times.95$. That would be the probability that they BOTH keep working – not what was asked for.
SANITY CHECK: Hello! If you have 2 calculators, the chance at least 1 stays alive, is at least as great as the chance that one particular one alone stays alive. I.e. you are better off than if you'd just brought one of the two calculators! So answer *must* exceed 0.98.

14. Out of 5378 people who drove to work yesterday, 7 crashed. Also, 11 were drunk. What is the probability a random driver was BOTH drunk AND crashed?
ANS e. cannot determine from information given.
Drunkness and crashing are *dependent* events. So cannot use AND-multiply rule. We are not told emphhow they depend on each other, there is no way to tell the answer!

15. In the preceding question, what is the probability a random driver was drunk OR, when you flipped a coin, you got heads? (rounded off to 7 or more decimals...)
ANS. a=.5010226850.
Coin and drunkness are independent events, so can use AND-multiply rule. Get prob(Drunk AND heads)= $(1/2) \times (11/5378)$. But... careful... we want prob(Drunk OR heads). These events not disjoint (since both can happen at same time) so must use the *non*disjoint OR rule, to get as our final ans $11/5378 + 1/2 - (1/2) \times (11/5378)$ (the final term subtracts off the twice-counted overlap so that it is correctly only once-counted). This is $5389/10756 = .5010226850$.

16. If you pick a card from a (52-card) pack and roll a dice, what is the probability you get an ace (there are 4 aces in a pack) AND you get a roll of 3?
ANS a= $1/(13 \times 6) = \frac{4}{52} \times \frac{1}{6}$. Card and dice are Independent events (no way one can affect the other), so use AND-multiply rule, get $(1/13) \times (1/6)$. Note $4/52 = 1/13$, and please do fractions correctly: $(A/B) \times (C/D) = (AC)/(BD)$.

17. I roll a dice and flip a coin. If the dice comes up 1 AND the coin tails, then I win a bet and get \$X. Otherwise I lose \$1. In order to make the bet "fair" what should X be?
ANS. b: $X = 11$
Prob of both is $(1/6) \times (1/2) = 1/12$ by AND-multiply rule for independent events.
So odds are $1/12 : 11/12$. So *fair* bet is 1:11 odds.
POPULAR WRONG ANS: $X = 12$. Nope: expected bet winnings would be $\frac{1}{12}12 + \frac{11}{12}(-1) = \frac{1}{12} \neq 0$. So that would be a *biased* bet in my favor. Now if I could just collect this money from all the students who did it wrong...

18. On an 18-question test, say each question is a 5-choice multiple choice question and each question has exactly ONE right answer (unlike this test, which contains a "joker" question with 2 right answers!). If you are clueless and guess randomly on all questions – how many right answers will you get *in expectation*?
ANS c = $18/5 = 3.6$. Expect to get 1/5 right answer per question, and 18 questions. So $\frac{1}{5} \times 18$ by adding up the expectations for all the 18 questions.

19. **List of useful formulas.**
(A) n choose r is ${}_nC_r = \frac{n!}{(n-r)!r!} = \frac{n(n-1)(n-2)\ldots(n+1-r)}{r(r-1)(r-2)\ldots1}$
(B) n perm r is ${}_nP_r = \frac{n!}{(n-r)!} = n(n-1)(n-2)\ldots(n+1-r)$
(C) In exponential growth with doubling time T_{doub}, the amount of stuff after time T is $2^{T/T_{doub}}Q_0$ where Q_0 is the amount of stuff you started with.
(D) In exponential decay with halving time T_{half}, the amount of stuff after time T is $2^{-T/T_{half}}Q_0$ where Q_0 is the amount of stuff you started with.
(E) Equation of a line is $y = sx + k$ where s =rise/run is the "slope" of the line and k is the y-intercept of the line (height where it crosses the y-axis).
(F) To solve $A^B = C$ for B, answer is $B = log(C)/log(A)$.
(G) To solve $A^B = C$ for A, answer is $A = C^{1/B}$.
(H) $n! = 1 \times 2 \times 3 \times \cdots \times n$ is the number of orderings of n things. (Pronounced "'n factorial.")

MORE useful formulas (YOU SHOULD KNOW THESE WITHOUT ME TELLING YOU).
Odds of an event: the ratio Prob(Event happening) : Prob(Not happening). Yields fair bets (zero expected win or loss).
Prob(a AND b) = Prob(a)×Prob(b) if a, b *independent* (i.e. cannot affect the chances of the other).
Prob(a AND b) = Prob(a)×Prob(b given that a happened) if b depends on a.

Prob(a OR b) = Prob(a)+Prob(b) if a, b *disjoint*, i.e. if only one of them can happen, not both.
Prob(a OR b) = Prob(a)+Prob(b)−Prob(a AND b) valid even if a and b non-disjoint, i.e. even if both can happen at the same time.
Prob(NOT a) = 1-Prob(a). Always true.
Prob(x HAPPENS AT LEAST ONCE IN n TRIES) = 1 - Prob(x does not happen on some single try)^n, if all tries independent.
Expectation value of something called v = {Sum, over all possible outcomes v, of Prob(v) $\times$ v.}
Expectation value in N repeated experiments (all same) = $N\times$(expected value in just one experiment).