↑name↑ ↑ID#↑ May 13, 2003. (3 pages.)

Math 127(3) – Spr 2003 – Final – Warren D. Smith.

1-page "crib sheet" allowed. Calculators allowed. Books and notes NOT allowed. Show all work. Some consecutive problems are referring to the SAME stuff hence are really just continuations of the same problem. Namely these groups {1,2,3,4}, {5,6,7,8}, {9,10,11,12,13}, {15,16}, {17,18}, {20,21,22}, {25,26,27} really can be thought of as just multiple parts of a single problem.

1. You have a parallelogram in 3-dimensional space PQRS (in order around its boundary; sides are P-Q, Q-R, R-S, and S-P). Coordinates of 3 of the vertices: $\vec{P} = (-1, 2, 3)$; $\vec{Q} = (2, 0, 1)$; $\vec{S} = (1, 3, 6)$. What are the coordinates of $\vec{R}$?
   ANS. $\vec{P} + \vec{R} = \vec{Q} + \vec{S}$ so $\vec{R} = \vec{Q} + \vec{S} - \vec{P} = (1 + 2 + 1,\ 3 + 0 - 2,\ 6 + 1 - 3)$
   $= (4,\ 1,\ 4)$.

2. (same parallelogram) What is the parallelogram angle θ at P? [Write as an exact formula involving an inverse-trig function, such as $\theta = \arccos(17/\sqrt{53})$.]
   ANS. The two edges going out of vertex P are $\vec{e} = \vec{Q} - \vec{P} = (3, -2, -2)$ and $\vec{f} = \vec{S} - \vec{P} = (2, 1, 3)$. Then $\theta = \arccos \frac{\vec{e}\cdot\vec{f}}{|\vec{e}|\,|\vec{f}|}$. The ingredients we need are $\vec{e}\cdot\vec{f} = 6 - 2 - 6 = -2$, $|\vec{e}| = \sqrt{9 + 4 + 4} = \sqrt{17}$, and $|\vec{f}| = \sqrt{4 + 1 + 9} = \sqrt{14}$. So $\theta = \arccos \frac{-2}{\sqrt{14 \times 17}}$
   $\theta = \arccos \frac{-2}{\sqrt{238}}$.

3. (same parallelogram) What is parallelogram's area?
   ANS. $\vec{e} \times \vec{f} = (-4,\ -13,\ 7)$, so the area$= |\vec{e} \times \vec{f}| = \sqrt{4^2 + 13^2 + 7^2} = \sqrt{234} = 3\sqrt{26} \approx 15.297$.

4. (same parallelogram) What is equation obeyed by the (x, y, z) that lie in the *same plane* as the parallelogram? (I.e. what is the equation of this plane?)
   ANS. The equation is $(x, y, z) \cdot (-4,\ -13,\ 7) = c$ where since we know $\vec{Q}$ is on the plane,
   $c = \vec{Q} \cdot (-4,\ -13,\ 7)$. [One could also have used $c = \vec{P} \cdot (-4,\ -13,\ 7)$ since $\vec{P}$ is also on the plane. The equality of these two c's is a good sanity check.] We have $c = -8 + 0 + 7 = -1$ from $\vec{Q} = (2, 0, 1)$. Sanity: We also have $c = 4 - 26 + 21 = -1$ from $\vec{P} = (-1, 2, 3)$. So final answer is (I negated both sides to make it cleaner) is:
   $(x, y, z) \cdot (4,\ 13,\ -7) = 1$ or $4x + 13y - 7z = 1$.

5. Here is a curve parameterized by t: $x = t\sin(3 - 3t)$, $y = 5e^{4t-4}$, $z = 2t$. What is a formula for the velocity vector (which is a tangent vector) to this curve?
   ANS. veloc$= (\dot{x}, \dot{y}, \dot{z}) = (\sin(3 - 3t) - 3t\cos(3 - 3t),\ 20e^{4t-4},\ 2)$.

6. (same curve) When $t = 1$, what is that velocity vector?
   ANS. veloc$= (\sin(0) - 3\cos(0),\ 20e^0,\ 2) = (-3,\ 20,\ 2)$.

7. (same curve) What is a parameterized equation for the line that is tangent to the curve at the point where $t = 1$?
   ANS. This point (where $t = 1$) is $(x, y, z) = (0, 5, 2)$. So the line going thru this point in the direction $(-3, 20, 2)$ is $(x, y, z) = (-3, 20, 2)s + (0, 5, 2)$ for all real values of the parameter s.

8. (same curve) What is the unit-norm (i.e. unit-length) version of the tangent vector to the curve at $t = 1$?
   ANS. The unit normalized version of $(-3, 20, 2)$ is $(-3, 20, 2)/\sqrt{413}$.

9. Here is a function: $F(x, y, z) = \ln(x^2 + 8y^2 - 4yz)$. What is its gradient $\vec{\nabla}F$?

   ANS. $\vec{\nabla}F = \frac{(2x,\ 16y-4z,\ -4y)}{x^2+8y^2-4yz}$. This is just because $\frac{d}{dq}\ln(H(q)) = H'(q)/H(q)$ from the chain rule and $\ln' x = 1/x$.

10. (same function F.) What is the value of that gradient $\vec{\nabla} F$ at the point $(x, y, z) = (1, 1, 2)$?
ANS. $\vec{\nabla} F = (2,\ 8,\ -4)$.

11. (same function F.) What is the directional derivative of F [at the same point $(x, y, z) = (1, 1, 2)$] in the direction $(3, -5, 2)/\sqrt{38}$?
ANS. $(2,\ 8,\ -4) \cdot (3, -5, 2)/\sqrt{38} = (6 - 40 - 8)/\sqrt{38} = -42/\sqrt{38}$.

12. (same function F.) Consider the level surface $F(x, y, z) = 0$. What direction is normal to this surface at the point $(1, 1, 2)$?
ANS. $\vec{\nabla} F = (2,\ 8,\ -4)$ since gradients are normal to level surfaces!

13. (same surface $F = 0$.) What is the equation of the tangent plane to this surface, with the point of tangency being $(1, 1, 2)$?
ANS. $(2,\ 8,\ -4) \cdot (x, y, z) = c$ where $c = (2, 8, -4) \cdot (1, 1, 2) = 2 + 8 - 8 = 2$ sicne we know $(1, 1, 2)$ is on the plane. So final ans is $2x + 8y - 4z = 2$.

14. Here is a plane: $x + 2y - z = 9$. Find the point on this plane that is closest to the point $(1, -7, 5)$. [Hint: Lagrange multiplier trick may help.] What is this minimal distance?
ANS. Squared Distance function is $F(x, y, z) = (x - 1)^2 + (y + 7)^2 + (z - 5)^2$. We want to minimize this subject to the constraint $x + 2y - z = 9$. So Lagrange trick: make $\vec{\nabla} F$ be parallel to the gradient $(1, 2, -1)$ of the constraint function. So $2x - 2 = k$, $2y + 14 = 2k$, $2z - 10 = -k$. Solving gives $x = (2 + k)/2 = 1 + k/2$, $y = k - 7$, $z = (10 - k)/2 = 5 - k/2$. Choosing k so these x, y, z actually lie on the plane means making $(2 + k)/2 + 2k - 14 - (10 - k)/2 = 9$ $3k - 18 = 9$, $3k = 27$, $k = 9$. So... solution is $x = (2 + 9)/2$, $y = 9 - 7$, $z = (10 - 9)/2$, which is $x = 11/2$, $y = 2$, $z = 1/2$. Sanity check: $11/2 + 2 \times 2 - 1/2 = 9$ so this point really is on the plane. This distance from here to $(1, -7, 5)$ is dist$= |(9/2, 9, -9/2)| = \sqrt{(9/2)^2 + 9^2 + (9/2)^2} = \sqrt{81 + 81/2} = 9\sqrt{3/2}$. Note (as sanity check) this direction $(9/2, 9, -9/2)$ is perpendicular to the plane; which makes sense since the shortest distance from a point to a plane is perpendicular to that plane.

15. Here is a double integral: $I = \int_0^4 \int_{\sqrt{y}}^2 3\exp(x^3) \mathrm{d}x \mathrm{d}y$. Sketch the region of integration in the xy plane.

    ANS. Area beneath a parabola (opening upwards): $0 < y < x^2 < 4$ for $0 < x < 2$.

16. (Same double integral I.) Completely evaluate the double integral I. [Hint: Evaluating it straightforwardly seems impossible. But interchanging the order of integration – being careful about what happens to the bounds on the integrals when you do that... consult your sketch above – will convert it into something doable.]
ANS. $\int_0^2 \int_0^{x^2} 3\exp(x^3) \mathrm{d}y \mathrm{d}x = \int_0^2 3x^2 \exp(x^3) \mathrm{d}x = [\exp(x^3)]_0^2 = e^8 - 1$.

17. Here is a region R in the xy plane: $1 \leq x^2 + y^2 \leq 9$, $y \geq 0$. Sketch it.
ANS. Two concentric circles of radii 1 and 3 centered at $(0, 0)$, except only the top *semi*circles ($y > 0$) are used.

18. (Same region R.) Evaluate $\int\int (1 + 2x^2 + 2y^2) \mathrm{d}x \mathrm{d}y$ where the integral is over the region R from the previous problem. [Hint: polar coordinates help.]
ANS. This is $\int_0^{2\pi} \int_1^3 (1 + 2r^2) r \mathrm{d}r \mathrm{d}\theta$ $\int_0^{2\pi} \int_1^3 (r + 2r^3) \mathrm{d}r \mathrm{d}\theta = 2\pi[r^2/2 + 2r^4/4]_1^3 = 2\pi[r^2/2 + r^4/2]_1^3 = 2\pi 44$ $= 88\pi$.

19. The ball of radius 4 is $x^2 + y^2 + z^2 < 4^2$. Consider the region consisting of the points (x, y, z) that satisfy that condition (i.e. are in the ball) *and* $x > 0$, *and* $y > 0$, *and* $z > 0$, *and* $x < y$. What is the volume of that region? [Hint: spherical coordinates is one approach that may help.]
ANS. The volume of the whole sphere is $4\pi r^3/3 = 4\pi 4^3/3 = 256\pi/3$. The restrictions $x > 0$, $y > 0$, $z > 0$ means we are only in 1/8 of this sphere $[1/8 = (1/2) \times (1/2) \times (1/2)]$, namely the positive octant. The final restriction $x < y$ means we are only in 1/2 of that (the other half is $x > y$; these two halves have to be equal volume due to the total symmetry of the problem with x and y palying the same roles),

so the answer is BallVol/16, which is $16\pi/3$.
OTHER ANS. The region in spherical coords is $0 < r < 4$, $0 < \phi < \pi/2$, $0 < \theta < \pi/4$, as opposed to the whole sphere $0 < r < 4$, $0 < \phi < \pi$, $0 < \theta < 2\pi$, which is plainly 16 times larger. (Note the integrand $r^2 \sin(\phi)$ does not depend on θ and is split half and half by ϕ.)

20. Here is a vector field in the xy plane: $\vec{H}(x, y) = (y cos[xy] + 6,\ x\cos[xy] - 3)$. Verify that $\vec{H}$ is a gradient of some function $Q(x, y)$, that is $\vec{H}(x, y) = \vec{\nabla}Q$. Do this without actually computing $Q(x, y)$.
ANS. $\vec{\nabla} \times \vec{H} = (0, 0, 0)$ or (saving work) all we really need, since this is 2D, is the z-component of the curl, i.e. $\frac{d}{dy}(y\cos[xy] + 6) - \frac{d}{dx}(x\cos[xy] - 3) = 0$. This zeroness proves it's a gradient.

21. (Same $\vec{H}(x, y)$.) Now compute $Q(x, y)$ so that $\vec{H}(x, y) = \vec{\nabla}Q$.
ANS. $Q(x, y) = \sin(xy) + 6x - 3y + C$.

22. (Same $\vec{H}(x, y)$.) Evaluate the curve integral $\int \vec{H} \cdot \mathrm{d}\vec{s}$ over the curve $x = 4/t$, $y = t$ from $t = 1$ to $t = 4$. [Hint: Consider the Newton "conservation of energy" theorem.]
ANS. The start point of the curve is $(4, 1)$. The end point is $(1, 4)$. $Q(\text{end}) - Q(\text{start}) = Q(1, 4) - Q(4, 1)$ $= -6 - 24 + 3 = -27$.

23. Let C be the closed curve in the xy plane bounding the region $x^2 \le y \le 1$. Also define the vector field $\vec{F}(x, y) = (7y\sinh(x),\ xy^2)$ in the xy plane. What is the circulation-integral $\oint_C \vec{F} \cdot \mathrm{d}\vec{s}$ (around the curve anticlockwise)? [Hint: Green's theorem = Stokes's theorem is one approach that may help. Also it may help to know that $\sinh(-x) = -\sinh(x)$.]

ANS. The z-component of the curl is $\frac{d}{dy}(7y\sinh x) - \frac{d}{dx}(xy^2) = 7\sinh x - y^2$. Double-Integrating this within the parabolic region given (which is equal to this circulation integral by Stokes and/or Green's theorem) yields cancellation for the $\sinh x$ at positive and negative x, so we only need to worry about the $-y^2$, and $-\int_{-1}^{1}\int_{x^2}^{1} y^2 \mathrm{d}y\mathrm{d}x = -\int_{-1}^{1}[y^3/3]_{x^2}^{1}\mathrm{d}x = -\int_{-1}^{1}[1/3 - x^6/3]\mathrm{d}x = -2/3 + 2/21 = -4/7$.
OTHER ANS. $-\int_0^1\int_{-\sqrt{y}}^{+\sqrt{y}} y^2 \mathrm{d}x\mathrm{d}y = -\int_0^1 2y^{5/2}\mathrm{d}y = -[(4/7)y^{7/2}]_0^1\mathrm{d}y = -[(4/7)y^{7/2}]_0^1\mathrm{d}y = -4/7$.

24. Let T be some 3D region of volume 7. (It will not actually matter what region it is!) Suppose you know $\int\int\int_T (3e^{xyyz} - 10)\mathrm{d}x\mathrm{d}y\mathrm{d}z = 20$. Then what is $I = \int\int\int_T e^{xyyz}\mathrm{d}x\mathrm{d}y\mathrm{d}z$?

ANS. $3I - 10 \times 7 = 20$ so $3I = 90$ so $I = 90/3 = 30$.

25. Let S denote the interior of the 45°-rotated square in the xy plane whose 4 vertices are $(1, 0)$, $(0, 1)$, $(-1, 0)$, and $(0, -1)$. (It is not required that you sketch this square, but it may help.)
What is $\int\int_S (xe^y + ye^x)\mathrm{d}x\mathrm{d}y$?

ANS. 0. Because: The positive and negative x's cancel in the first term. The positive and negative y's cancel in the second term.

26. (Same square S.) What is $\int\int_S 5\mathrm{d}x\mathrm{d}y$? [Hint: what is the area of this square?]
ANS. Area of square is 2 since sidelength of square is $\sqrt{2}$. So ans is $5\times$area $= 5 \times 2 = 10$.

27. (Same square S.) Let $\vec{F}(x, y)$ be the following vector field: $\vec{F} = (3x,\ 2y + \arctan(x) + 37\cos(x)^2)$. What is the flux of $\vec{F}$ out of the square's boundary? [Hint: Gauss's divergence theorem may help.]

ANS. $\vec{\nabla} \cdot \vec{F} = 3 + 2 = 5$. So by the divergence theorem the flux is $\int\int 5\mathrm{d}x\mathrm{d}y = 5 \times 2 = 10$. Same double integral and same answer as last problem.