↑name↑ ↑ID#↑ April 29, 2003. (2 pages.)

Math 127(3) – Spr 2002 – Qz2 ANSWERS – Warren D. Smith.

1-page "crib sheet" allowed. Calculators allowed. Books and notes NOT allowed. **170/170=perfect.**

1. Find the length of the curve $(x, y, z) = (4t, 3t + 1, 5\cosh t)$ from $t = 0$ to $t = 1$. [Hint. The following facts may be useful: $\frac{d}{dx}\cosh x = \sinh x$, $\frac{d}{dx}\sinh x = \cosh x$, and $1 + \sinh^2 x = \cosh^2 x$.]
   ANS. $\int_0^1 \sqrt{4^2 + 3^2 + 5^2 \sinh^2 t}\,dt = \int_0^1 \sqrt{25 + 25\sinh^2 t}\,dt = 5\int_0^1 \sqrt{1 + \sinh^2 t}\,dt = 5\int_0^1 \cosh t\,dt = 5[\sinh t]_0^1$ $= 5\sinh 1 \approx 5.8760$.

2. Find a parameterization of the curve $\tan(y) + x^2 = 1$.
   ANS. Simplest is to use x as the parameter. $y = \arctan(1 - x^2)$ and $x = x$, for $-\infty \le x \le \infty$.
   WORSE ANS: $x = \sqrt{1 - \tan y}$ and $y = y$ for $-\pi/4 \le y \le \pi/4$ because this only gives the $x > 0$ half of the curve. You then would also need to do $x = -\sqrt{1 - \tan y}$ and $y = y$ for $-\pi/4 \le y \le \pi/4$ to get the other half of the curve with $x < 0$.
   BETTER ANS (WORTH EXTRA CREDIT): Actually, $y = n\pi + \arctan(1 - x^2)$ and $x = x$, for $-\infty \le x \le \infty$ and n any integer, is the *really* great answer, since my first ans only parameterized one of these curves ($n = 0$) but in fact there are an infinite number of them... (all integer n). Also in the "worse ans" it was for the same reason better to say "$tany \le 1$" instead of "$-\pi/4 \le y \le \pi/4$."

3. Let $\vec{F}(x, y, z) = (x^2,\ xz,\ y^2 z)$. What is its curl $\vec{\nabla} \times \vec{F}$?
   ANS $(2yz - x,\ 0,\ z)$. Using defn of curl $\vec{\nabla} \times \vec{F} = (\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial x},\ \frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z},\ \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y})$. Here $F_1 = x^2$, $F_2 = xz$, $F_3 = y^2 z$. Result is $(2yz - x,\ 0 - 0,\ z - 0)$, a VECTOR.

4. And what is its divergence $\vec{\nabla} \cdot \vec{F}$?
   ANS. $2x + 0 + y^2$. Using defn of divergence $\vec{\nabla} \cdot \vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$, a SCALAR.

5. What is the curve integral $\int \vec{F} \cdot d\vec{s}$ along the curve $x = t$, $y = t^2$, $z = t^3$ for $t = 0$ to $t = 1$? Express as an ordinary integral and then do it.
   ANS. $d\vec{s} = \frac{d}{dt}(x, y, z) = \vec{veloc}dt = (1,\ 2t,\ 3t^2)dt$. And $\vec{F}$ expressed in terms of t is $(t^2,\ t^4,\ t^7)$. So ans is $\int_0^1 (t^2,\ t^4,\ t^7) \cdot (1,\ 2t,\ 3t^2)dt = \int_0^1 [t^2 + 2t^5 + 3t^9]dt = [t^3/3 + 2t^6/6 + 3t^{10}/10]_0^1 = [1/3 + 1/3 + 3/10]$ $= 20/30 + 9/30 = 29/30$.

6. Consider the surface $\sin(x) + y^2 + z = 1$. What is a unit normal vector to this surface at a point (x, y, z) on it?
   ANS. Gradient of F is normal to level surface $F = const$, so, ans is $(\cos x,\ 2y,\ 1)/\sqrt{\cos(x)^2 + 4y^2 + 1}$.

7. What would be a parameterization of that surface?
   ANS. Use x and y as parameters. $x = x$, $y = y$, $z = 1 - \sin(x) - y^2$, for all real x,y.

8. What would be the infinitesimal element of scalar-area for your parameterization? (That is, if your parameters were p and q, the scalar-area element would be $F(p, q)dpdq$ for soem function F, which you would have to find.)
   ANS. In this case, since we have $z(x, y)$ as an explicit height function, we may use the simple formula $\sqrt{1 + (\frac{dz}{dx})^2 + (\frac{dz}{dy})^2}\,,dxdy$ for the area element. So ans is $\sqrt{1 + 4y^2 + \cos(x)^2}\ dxdy$. If you had given a parameterization $(x, y, z) = \vec{r}(p, q)$ that was not in the convenient form of an explicit height function, then you'd have had to compute $|(\frac{d\vec{r}}{dp}) \times (\frac{d\vec{r}}{dq})|, dpdq$.

9. What would be the surface area of that surface where it intersects the cylindrical region $x^2 + y^2 \le 1$? Write fully explicitly as a double integral (with explicit bounds on the integral signs) but do not actually do either the inner or outer integral.

ANS. $\int_{x=-1}^{x=1}\int_{y=-\sqrt{1-x^2}}^{y=+\sqrt{1-x^2}} \sqrt{1+\cos(x)^2+(2y)^2}\ dy dx$. It is key here that the y integral be innermost because the bounds on that $\int$ depend on x.

10. Compute the Jacobian matrix of this change of variables:
$X(a,b,c) = a^2+b$, $Y(a,b,c) = a-3b$, $Z(a,b,c) = c$.
ANS.

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2a   1   0
1   -3   0
0    0   1
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11. So $dxdydz = K(a,b,c)dadbdc$, for what volume-adjustment function $K(a,b,c)$? (continuing previous question)
ANS. $K(a,b,c) = -6a-1 =$ determinant of the above matrix.

12. Evaluate $\int\int\int 7\, dxdydz$ over the region $1 \le x^2+y^2 < 4$, $0 < z < 3$.
ANS. Realize region is a height-3 cylinder of radius 2 with a coaxial cylindrical hole of radius 1. Ans is $7\times$ volume. And volume is $(\pi 2^2 - \pi 1^2)\times 3 = 9\pi$. So Ans is 63π.

13. Suppose, in the previous problem, the integrand, instead of being 7, had been $7+x$. How would that have affected the answer?
ANS. No change! It would not have affected the answer at all, because due to symmetry, the x is negative equally often as it is positive (East half versus West half of cylinder) so its integral is 0. [Note: the region (cylinder with hole) on integration does NOT change, only the quantity INSIDE the integral changes (to $7+x$). The point is $\int\int\int(7+x) = \int\int\int 7 + \int\int\int x$ and $\int\int\int x = 0$.] Easiest question on the test, and nobody got it???!!

14. Suppose $\vec{F}(x,y,z)$ is a function whose *curl* is $\vec{\nabla}\times\vec{F} = (2x,y,z)$. What is the curve-integral of $\int \vec{F}\cdot d\vec{s}$ around the unit circle curve $x^2+y^2=1$, $z=3$ (NOTE: $z=3$) going anticlockwise?
ANS. [Note: Since only know $\vec{\nabla}\times\vec{F}$, not $\vec{F}$ itself, cannot just do curve integral.] Use Stokes theorem to see it is the same as the surface integral of $\int\int(2x,y,z)\cdot d\vec{area}$ on the flat unit disk $x^2+y^2<1$, $z=3$. The outward unit normal vector $\vec{u}$ to the disk is $(0,0,1)$. (NOTE: that makes it easy. Picking a hemisphere would have made life considerably harder... you can pick any surface bounded by this unit circle, so pick one that makes lfe easy.) Since by definition $d\vec{area} = \vec{u}darea$, and since $(2x,y,z)\cdot(0,0,1) = z$, this integral is $\int\int z darea$ over the disk $x^2+y^2<1$, $z=3$. Since $z=3$ this is $= 3area = 3\pi$.

OTHER ANS. Since $\vec{\nabla}\cdot\vec{F} = 2+1+1 = 4 \ne 0$, this $\vec{F}$ cannot *be* a curl of anything (since div of curl is 0 always)!!!

15. What is the flux of $(5x, y+x^2, z)$ upward through the circular disk $x^2+y^2\le 1$, $z=3$?
ANS. Comes out to be the same integral as last problem! Thought this was going to give you a clue, and/or make you have an easy time on *one* of these problems... The flux surface integral is $\int\int(5x,\ y+x^2,\ z)\cdot d\vec{area}$ on the unit disk $x^2+y^2<1$, $z=3$. Since the outward unit normal vector $\vec{u}$ to the disk is $(0,0,1)$, and since by definition $d\vec{area} = \vec{u}darea$, and since $(5x,\ y+x^2,\ z)\cdot(0,0,1) = z$, this integral is $\int\int z darea$ over the disk $x^2+y^2<1$, $z=3$. Since $z=3$ this is $= 3area = 3\pi$. [Note: can't use divergence theorem to re-express as a volume integral, since no volume is *enclosed* by a flat disk! Divergence thm only applicable when computing flux out of a surface that completely encloses something; here we have only an incomplete surface.]

16. Find the flux of $\vec{F}(x,y,z) = (5x, y+x^2, z)$ out of the whole cylinder (*including* both its flat circular endcaps and the curved part) $x^2+y^2\le 1$, $0\le z\le 3$.
ANS. Using the divergence theorem and the fact $\vec{\nabla}\cdot\vec{F} = 5+1+1 = 7$, we get that the flux is $\int\int\int 7dxdydz = 7volume = 7\cdot 3\pi = 21\pi$.

OTHER APPROACH. Without using divergence thm, you'd have trouble doing the flux integral $\int\int(5x, y+x^2, z)\cdot(x,y,0)d\,area = \int\int(5x^2+y^2+yx^2+0)d\,area$ over the curved part of the cylinder. At this point you'd have to convert to cylindrical coords and get some trig integrals. It could be done, then you'd have to add that to your previous top-endcap flux result 3π, and the bottom endcap flux result 0.

17. Find the surface (double) integral of x^2+y^2 over the flat unit disk $x^2+y^2 \leq 1$ in the xy plane. In other words I want to know the numerical value of $\int\int_{\bullet}(x^2+y^2)\mathrm{d}x\mathrm{d}y$. [Hint. Try polar coordinates.]
ANS. Remembering that $r^2 = x^2+y^2$ and that the area-adjustment Jacobian factor for polar coordinates is r, i.e. that the correct $d$area in polar coords is $r\mathrm{d}r\mathrm{d}\theta$ (and not $\mathrm{d}r\mathrm{d}\theta$) we get $\int_{r=0}^{r=1}\int_{\theta=0}^{\theta=2\pi} r^2\ r\mathrm{d}\theta\mathrm{d}r$ $= \int_{r=0}^{r=1} 2\pi r^2 r\ \mathrm{d}r = \int_{r=0}^{r=1} 2\pi r^3\ \mathrm{d}r = [2\pi r^4/4]_0^1 = \pi/2$.

   HARDER WAY: You could have stayed with Cartesian coordinates and written $\int_{x=-1}^{x=1}\int_{y=-\sqrt{1-x^2}}^{+\sqrt{1-x^2}}(x^2+y^2)\mathrm{d}y\mathrm{d}x$. Do the inner integral to get $\int_{x=-1}^{x=1}\left[x^2y+y^3/2\right]_{y=-\sqrt{1-x^2}}^{+\sqrt{1-x^2}}\mathrm{d}x = \int_{x=-1}^{x=1}\left[2x^2\sqrt{1-x^2}+(1-x^2)^{3/2}/2\right]\mathrm{d}x$ and now doing the outer integral is difficult.