↑name↑ ↑ID#↑ May 1, 2003. (3 pages.)

Math 127(3) – Spr 2002 – Quiz 2 – Warren D. Smith.

1-page "crib sheet" allowed. Calculators allowed. Books and notes NOT allowed. Perfect 170/170.

1. True or false: If $\vec{F}$ is a vector field, then div$\vec{F}$ is a vector field.
   ANS: False, it is scalar.

2. True or false: If $\vec{F}$ is a vector field, then curl$\vec{F}$ is a vector field.
   ANS: True.

3. True or false: $\int_C \vec{\nabla} F(x,y,z) \cdot d\vec{s} = 0$ if $F(x,y,z)$ is smooth and C is a smooth closed curve (by "closed" I mean, such as a circle, it closes upon itself, start point same as end point).
   ANS: True. The curl of a grad is 0, so by Stokes's theorem the curve integral is $\int\int 0 = 0$. Also Newton Conservation of Energy theorem gives 0 since start and end points are same, and integrand is a gradient (energy−same energy= 0).

4. If $\vec{F} = (x + y^2, \arctan(z) - y, e^x)$ then $\int_S \vec{F} \cdot \mathrm{d}\vec{\mathrm{area}} = 0$ where S is the surface of a sphere.
   ANS: true. The divergence of $\vec{F}$ is $1 - 1 + 0 = 0$ so by the divergence theorem any such flux integral must be 0.

5. There is a vector field $\vec{F}$ such that curl$\vec{F} = (x, y, z)$.
   ANS: False. Because div$\vec{F} = 1 + 1 + 1 = 3 \neq 0$, and div of a curl is always 0, we know that (x, y, z) is not a curl of any function.

6. What is the flux of $\vec{F} = (2x^2, y, z^2)$ across the rectangle described by the following inequalities and equalities: $0 < x < 1$, $0 < z + y < 2$, $z = y$?
   WRONG ANS: You CANNOT use the divergence theorem here because this surface does not enclose any volume! It is just a flat rectangular plate!
   ANS: Parameterized(x, y) surface is $x = x$, $y = y$, $z = y$, $0 < x < 1$ and $0 < y < 1$. Unit outward normal (well, I did not say what "outward" was, so may pick either sign) is $(0, -1, 1)/\sqrt{2}$; since surface was level set of $z - y$, gradient= $(0, -1, 1)$ is normal to surface, and the $\sqrt{2}$ normalizes it to unit length. We have darea= $\sqrt{2}\mathrm{d}x\mathrm{d}y$ where (since parameters are x and y and have a height function) $\sqrt{2} = \sqrt{1 + (\partial z/\partial x)^2 + (\partial z/\partial y)^2} = \sqrt{1 + 1 + 0}$ here. So putting all the ingredients together (notice the $\sqrt{2}$'s on the top and bottom cancel)... we have $\int_0^1 \int_0^1 (2x^2, y, y^2) \cdot (0, -1, 1)\mathrm{d}x\mathrm{d}y = \int_0^1 \int_0^1 (-y + y^2)\mathrm{d}x\mathrm{d}y$ $= \int_0^1 (-y + y^2)\mathrm{d}y = [(-y^2/2 + y^3/3]_0^1 = (1/3 - 1/2) = -1/6$. (or $+1/6$ would also be fine.)

7. Find the surface area of the part of the conical surface $z = \sqrt{x^2 + y^2}$ that lies inside the infinite-square prism $|x| < 1$, $|y| < 1$.
   ANS: Since height function, darea= $\sqrt{1 + (\partial z/\partial x)^2 + (\partial z/\partial y)^2}\,\mathrm{d}x\mathrm{d}y$. We have $\partial z/\partial x = x/\sqrt{x^2 + y^2}$ and $\partial z/\partial y = y/\sqrt{x^2 + y^2}$ and so darea= $\sqrt{1 + (x^2 + y^2)/(x^2 + y^2)} = \sqrt{1 + 1} = \sqrt{2}$. (Note: "roof-slant" slope everywhere is 1, so area-adjust factor is $\sqrt{1}$.) So surface area is $\int_{-1}^1 \int_{-1}^1 \sqrt{2}\mathrm{d}x\mathrm{d}y = 2 \times 2 \times \sqrt{2} = 4\sqrt{2}$.

8. Find a parameterization of the curve $\mathrm{arcsinh}(y) + x^2 = 1$.
   ANS. $y = \sinh(1 - x^2)$ for all real x.
   The above is a BETTER answer than $x = \pm\sqrt{1 - \mathrm{arcsinh}(y)}$ for $y < \sinh 1$ since the conditions are easier and don't need to glue two curves together.

9. Let $\vec{F}(x, y, z) = (x,\ xz,\ y^2z^2)$. What is its curl $\vec{\nabla} \times \vec{F}$?
   ANS. $(2\,yz^2 - x,\ 0 - 0,\ z - 0) = (2\,yz^2 - x,\ 0,\ z)$.

10. (Same $\vec{F}$.) And what is its divergence $\vec{\nabla} \cdot \vec{F}$?
    ANS. $1 + 0 + 2y^2z = 1 + 2y^2z$.

11. (Same $\vec{F}$.) What is the curve integral $\int \vec{F}\cdot \mathrm{d}\vec{s}$ along the curve $x=t^3$, $y=t^2$, $z=t$ for $t=0$ to $t=1$? Express as an ordinary integral and then do it.
WRONG ANS. Trying to use Stokes theorem to convert to a surface integral is insane because this is not a closed curve (i.e., it has 2 different endpoints) and hence cannot be the bounding curve of any surface!
ANS. $\int_0^1 (t^3,t^4,t^6)\cdot(3t^2,2t,1)\mathrm{d}t = \int_0^1(3t^5+2t^5+t^6)\mathrm{d}t = \int_0^1(5t^5+t^6)\mathrm{d}t = [6t^6/6+t^7/7]_0^1 = 5/6+1/7$ $= 41/42$.

12. How would you describe the following region: a sphere of radius 3, centered at $(0,0,0)$ having a cylindrical hole of radius 2 bored through its center in the z-axis direction, as a set of inequalities among x,y,z?
ANS: $x^2+y^2+z^2\le 9$ and $x^2+y^2\ge 4$.

13. (same region as previous problem) What would $\int\int\int x\mathrm{d}x\mathrm{d}y\mathrm{d}z$ be, over the sphere-with-hole region of the previous problem?
ANS. 0 by symmetry.

14. (same region as previous problem) What would $\int\int\int 1\mathrm{d}x\mathrm{d}y\mathrm{d}z$ be, over the sphere-with-hole region of the previous problem?
ALMOST RIGHT ANS. This is the volume of the region. The volume of the sphere is $4\pi r^3/3 = 4\pi 3^3/3 = 4\pi 9 = 36\pi$. The volume of the hole (which we need to subtract off) is $\pi r^2 h$ where $r=2$. What is the height h? We know $x^2+y^2=4$ and $x^2+y^2+z^2=9$ where the hole hits the sphere surface so that $z^2=9-4=5$ there. So $z=\pm\sqrt{5}$. So $h=2\sqrt{5}$. So final answer is $36\pi - 8\pi\sqrt{5}\approx 56.89$.
RIGHT ANS. Unfortunately, the above ans (which was my original intent) had forgotten to deal with the two "curved endcap" pieces $x^2+y^2\le 4$, $x^2+y^2+z^2<9$ with $|z|>\sqrt{5}$. These each have volume $\pi\int_{\sqrt{5}}^3(9-z^2)\mathrm{d}z = \pi[9z-z^3/3]_{\sqrt{5}}^3$ $(27-9-9\sqrt{5}+5\sqrt{5}/3)\pi = (18-22\sqrt{5}/3)\pi \approx 5.03$. So the *correct* answer, after subtracting off these two pieces also, is $36\pi - 8\pi\sqrt{5} - 36\pi + 44\pi\sqrt{5}/3 = 20\pi\sqrt{5}/3\approx 46.83$.

15. Suppose $\vec{F}(x,y,z)$ is a function whose *curl* is $\vec{\nabla}\times\vec{F} = (2x,-y,-z)$. What is the curve-integral of $\int \vec{F}\cdot\mathrm{d}\vec{s}$ around the unit square curve $0\le x\le 1$, $0\le y\le 1$, $z=1$ going anticlockwise?
ANS. Use Stokes thm. Get $\int_0^1\int_0^1(2x,\ -y,\ -z)\cdot(0,0,1)\mathrm{d}x\mathrm{d}y = \int_0^1\int_0^1 -z\mathrm{d}x\mathrm{d}y = (-1)\mathrm{area} = -1$. Note, Stokes theorem involves the "right hand rule:" if your right hand's curled fingers are the direction around the curve, your thumb points toward the "outward" normal to the surface.

16. Find the surface (double) integral of $(x^2+y^2)^2$ over the flat unit disk $x^2+y^2\le 1$ in the xy plane. In other words I want to know the numerical value of $\int\int_{\bullet}(x^2+y^2)^2\mathrm{d}x\mathrm{d}y$. [Hint. Polar coordinates might help.]
ANS. $\int_0^{2\pi}\int_0^1(r^2)^2 r\mathrm{d}r\mathrm{d}\theta$ do the θ integral first, just multiplies us by 2π. Then $= 2\pi\int_0^1 r^5\mathrm{d}r = 2\pi[r^6/6]_0^1 = 2\pi/6 = \pi/3$.

17. Suppose $\vec{F}(x,y,z)$ is the gradient of $\tan x$. What is the integral of $\vec{F}\cdot\mathrm{d}\vec{s}$ along *any* curve from $(0,1,5)$ to $(\pi/4,6,2)$?
ANS. Use Newton theorem that $\int(\vec{\nabla}F)\cdot\mathrm{d}\vec{s} = F(\vec{B})-F(\vec{A})$ where th eintegral is along any curve from $\vec{A}$ to $\vec{B}$. Get $\tan(\pi/4)-\tan(0)=1$.
HARDER WAY. Find $\tan' x = \sec(x)^2$, parameterize a curve (easiest is a line segment, presumably, e.g. $x=\pi t/4$, $y=5t+1$, $z=5-3t$ for $0<t<1$), and do the integral, e.g. $\int_0^1(\sec^2 x,0,0)\cdot(\pi/4,5,-3)\mathrm{d}t$ $= \int_0^1(\pi/4)\sec^2 x\mathrm{d}t = \int_0^1(\pi/4)\sec^2(\pi t/4)\mathrm{d}t = \int_0^{\pi/4}\sec^2 x\mathrm{d}x = [\tan x]_0^{\pi/4} = 1$.