↑name↑ ↑ID#↑ February 20, 2003. Show all work.

Math 127 – Spr 2002 – Quiz 1 ANSWERS – Warren D. Smith

1. In the below group of questions (up to and including #8), let $\vec{a} = (4, 4, 2)$ and $\vec{b} = (2, -1, 7)$. Compute: $|\vec{a}|$. ANS $\sqrt{4^2 + 4^2 + 2^2} = \sqrt{36} = 6$

2. $|\vec{b}|$, ANS $\sqrt{2^2 + 1^2 + 7^2} = \sqrt{54} = 3\sqrt{6}$

3. The unit-length version $\hat{a}$ of $\vec{a}$, ANS $\hat{a} = \frac{(4,4,2)}{6} = \frac{(2,2,1)}{3}$

4. $\vec{a} \cdot \vec{b}$, ANS $18 = 4 \cdot 2 - 4 + 2 \cdot 7$

5. the distance between $\vec{a}$ and $\vec{b}$. ANS $|\vec{a} - \vec{b}| = |(2, 5, -5)| = \sqrt{2^2 + 5^2 + 5^2} = \sqrt{54} = 3\sqrt{6}$.

6. $\vec{a} \times \vec{b}$, ANS $(30, -24, -12)$. That in slow-mo: $\vec{a} \times \vec{b} = (a_2b_3 - a_3b_2,\ a_3b_1 - a_1b_3,\ a_1b_2 - a_2b_1) = (4 \cdot 7 + 2 \cdot 1,\ 2 \cdot 2 - 4 \cdot 7,\ -4 \cdot 1 - 4 \cdot 2)$. *Don't* screw up the signs. Recommended sanity checks: the dot products with $\vec{a}$ & with $\vec{b}$ are 0. The best way to remember how to do dot products (and 3×3 determinants) seems to be $\vec{a} \times \vec{b}$"=" $\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$ and then the determinant is the sum of 6 terms, each term being a product of 3 entries in the matrix lying on either a \ generalized diagonal (these 3 terms each get a − sign) or a / generalized diagonal (these 3 terms each get a + sign).

7. the angle between $\vec{a}$ and $\vec{b}$ (can write as simplified exact formula involving arccos or arcsin, but not necessary to compute a numerical approximation to that arccos or arcsin),
ANS $\arccos(\frac{\vec{a} \cdot \vec{b}}{|\vec{a}|\,|\vec{b}|}) = \arccos(\frac{18}{6\sqrt{54}}) = \arccos(\frac{1}{\sqrt{6}})$

8. the angle between the plane $\vec{a} \cdot (x, y, z) = 1$ and the other plane $\vec{b} \cdot (x, y, z) = 1$ (can write as simplified exact formula involving arccos or arcsin, but not necessary to compute a numerical approximation to that arccos or arcsin),

   ANS also $\arccos(\frac{1}{\sqrt{6}})$, since $\vec{a}$ and $\vec{b}$ are normal to their planes and the angle between planes is same as angle between their normal vectors. Note: some people prefer to use $90° - \theta$ instead of θ, which I must agree is equally legitimate. However in problem 7, really the θ formula given arguably is right and $90° - \theta$ is *not* because vectors are *oriented* quantities (have arrows – as opposed to undirected lines). If one were speaking of angles between *oriented* planes (each plane having a red side and a blue side) one could similarly disambiguate angle for planes.

9. Simplify:
$(2\vec{q} + \vec{p}) \cdot (\vec{p} \times \vec{q} + \vec{q} \times \vec{r} + \vec{p} \times \vec{r})$.
ANS First we distribute out and eliminate clearly-zero terms (dot prods of perpendicular vectors) to get

$$2\vec{q} \cdot (\vec{p} \times \vec{r}) + \vec{p} \cdot (\vec{q} \times \vec{r})$$

   but then we may go further by realizing these are both "vector triple products" as the book calls them, i.e. 3×3 determinants, $\vec{a} \cdot (\vec{b} \times \vec{c}) = \vec{b} \cdot (\vec{c} \times \vec{a}) = \vec{c} \cdot (\vec{a} \times \vec{b}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$ and so they are really both the *same* determinant (up to a factor of −2: remember a cyclic shift of the 3 letters leaves the vector triple product unchanged, but a swap of 2 letters changes its sign since $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$). So FINAL ANSWER: this is

$$\vec{r} \cdot (\vec{q} \times \vec{p})$$

.

10. Some subset of the following 8 formula-fragments involving 3-dimensional vectors, are clearly insane. Circle the insane ones and give some indication of what is crazy about them:
$(\vec{a}\cdot\vec{b})\vec{c}$, $\cos(\vec{a}\times\vec{b}*)\vec{c}$, $\cos(\vec{a}\cdot\vec{b})\vec{c}$, $\cos(\vec{a}\cdot\vec{b})*\times\vec{c}$, $\vec{a}\times\vec{b}+*\vec{a}\cdot\vec{c}$, $\vec{a}\cdot\vec{b}+*\vec{c}$, $\vec{a}\times\vec{b}+\vec{c}$, $(\vec{i}+\vec{j})*(\vec{i}+\vec{k})$.
ANS: I have starred the insane ones at roughly the places where they are insane due to use of a vector where a scalar is required, or vice versa. E.g. cos requires a *scalar* argument, can't take cross product of scalar with vector, can't add scalar to vector, etc. The last one tries to "multiply" two vectors using a non-dot, non-cross (presumably scalar) multiply; that's insane [although I admit $(i+j)(i+k)$ would have a legitimate meaning if i, j, k were regarded as *quaternions* rather than vectors. I put $\vec{i}$ arrows so you can't weasel out of it that way.]

11. In this and the next group of questions (up to and including #13), let

$$F(x, y, z) = \frac{1}{(x-y)^2+1} - (x+y+z)^2 + 2\sin z.$$

What is its gradient $\vec{\nabla}F(x, y, z)$? (Hints. (a) Answer may be neatly written as 2 times a vector-valued-formula not involving any factors of 2. (b) the t-derivative of $\frac{1}{t^2+1}$ is $\frac{-2t}{(t^2+1)^2}$.) ANS:

$$2\left(-\frac{x-y}{\left((x-y)^2+1\right)^2} - x - y - z,\ \frac{x-y}{\left((x-y)^2+1\right)^2} - x - y - z,\ -x-y-z+\cos(z)\right)$$

That in slow-mo: $\vec{\nabla}F(x, y, z) = (\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z})$ by definition. Now, when we take those partial derivatives, we use $\frac{\partial}{\partial x}[(x+y+z)^2] = 2(x+y+z)(1+0+0) = 2(x+y+z) = \frac{\partial}{\partial y}[(x+y+z)^2] = \frac{\partial}{\partial z}[(x+y+z)^2]$ where note y and z are regarded as *constants* when taking the x-partial, etc. And we use $\frac{\partial}{\partial y}[\frac{1}{(x-y)^2+1}]$ is $-[\frac{-2(x-y)}{([x-y]^2+1)^2}]$ where note the $-$ sign because we are taking the t derivative where $t = x - y$ and then $\frac{\mathrm{d}t}{\mathrm{d}y} = -1$. Similarly $\frac{\partial}{\partial y}[\frac{1}{(x-y)^2+1}] = \frac{-2(x-y)}{([x-y]^2+1)^2}$, now note + sign.

12. What is the rate of increase of $F(x, y, z)$ in the direction $\hat{a}$ from question 3?
ANS:

$$\hat{a}\cdot\vec{\nabla}F = \frac{2}{3}\left(2[\frac{y-x}{\left((x-y)^2+1\right)^2} - x - y - z] + 2[\frac{x-y}{\left((x-y)^2+1\right)^2} - x - y - z], + [-x-y-z+\cos(z)]\right)$$

$$= \frac{-10}{3}(x+y+z) + \frac{2}{3}\cos(z).$$

13. Find a local maximum of $F(x, y, z)$ (give the values of x, y, z and of F)? [**Hint.** There will be 3 major "chunks" of the equations you get for x, y, and z; what happens if you find x,y,z that cause each of these chunks to be zero?]

ANS. At the max (or at any flat spot) $\vec{\nabla}F = \vec{0}$. This is accomplished by $x = y = -\pi/4$ and $z = +\pi/2$. That way $x - y = 0$ and $x + y + z = 0$ and $\cos(z) = 0$ (these are the 3 major chunks I was hinting at) causing $\vec{\nabla}F = \vec{0}$ to be satisfied in all 3 components, and $\sin(z)$ is maximized. That makes it obvious this is a max since each term in the definition of F is maximized by itself. Here $F = 1 - 0 + 2 = 3$. (Another flat spot would gave been $x = y = +\pi/4$ and $z = -\pi/2$, but that would not have led to a max, since $\sin(z)$ is *min*imized at $z = -\pi/2$.)

14. **NOTE.** *Either* do this problem, or the previous problem, but not both, whichever you like better. Find a local minimum of $G(x, y, z) = 5x^2 + 6xy + 3y^2 + x - 7y + z^2 + 2z + 3$ (Give the values of x, y, z. Not necessary to give value of G.)

ANS: The gradient of G is $\vec{\nabla}G = (10x + 6y + 1,\ 6x + 6y - 7,\ 2z + 2)$. That in slo-mo: By definition $\vec{\nabla}G(x, y, z) = (\frac{\partial G}{\partial x}, \frac{\partial G}{\partial y}, \frac{\partial G}{\partial z})$. The first coordinate of $\vec{\nabla}G$ is the x-partial which is $\frac{\partial G}{\partial x}[5x^2 + 6xy + 3y^2 + x - 7y + z^2 + 2z + 3] = 10x + 6y + 0 + 1 - 0 + 0 + 0 + 0 = 10x + 6y + 1$. Now to find the min we set $\vec{\nabla}G = \vec{0}$ meaning we have to solve 3 equations $10x + 6y = -1,\ 6x + 6y = 7, 2z = -2$ for 3 unknowns x, y, z. Teh last eqn shows $z = -1$. Now helps to subtract 2nd eqn from 1st eqn, to solve for x easily; FINAL ANSWER is $x = -2, y = 19/6, z = -1$.

15. In this and the next group of questions (up to and including #16), let $\vec{F}(t)$ denote the parameterized curve $\vec{F}(t) = (3t + 2, 5\ln(\sec(t)), 4t + 7)$. If t means time, what is the velocity vector?

    ANS: $\vec{F}'(t) = (3, 5\tan(t), 4)$. Note $\frac{\mathrm{d}}{\mathrm{d}t}\ln\sec(t) = \frac{\sec'(t)}{\sec(t)}$ and $\sec'(t) = \sec(t)\tan(t)$ so that $\frac{\mathrm{d}}{\mathrm{d}t}[\ln\sec(t)] = \tan(t)$.

16. What is the arc length from $t = 0$ to $t = \pi/4$ of this curve? [Hint. Here is a short table of (the less well known) trig integrals, one or more of which may be useful to you:
    $\int \sec(x)\mathrm{d}x = \ln|\sec(x) + \tan(x)| + C$; $\int \csc(x)\mathrm{d}x = \ln|\csc(x) + \cot(x)| + C$;
    $\int \tan(x)\mathrm{d}x = \ln|\sec(x)| + C$; $\int \cot(x)\mathrm{d}x = \ln|\sin(x)| + C$.]

    ANS: Arclength= $\int_{t=0}^{t=\pi/4} |\vec{F}'(t)|\mathrm{d}t$ is the key formula, saying distance-traveled is |velocity| times time, where the integral adds up all the little chunks of time $\mathrm{d}t$. So Arclength= $\int_{t=0}^{t=\pi/4} |3, 5\tan(t), 4|\mathrm{d}t = \int_{t=0}^{t=\pi/4} \sqrt{9 + 16 + 25\tan(t)^2}\mathrm{d}t = \int_{t=0}^{t=\pi/4} 5\sqrt{1 + \tan(t)^2}\mathrm{d}t = 5\int_{t=0}^{t=\pi/4} \sqrt{\sec(t)^2}\mathrm{d}t = 5\int_{t=0}^{t=\pi/4} \sec(t)\mathrm{d}t = 5\ln|\sec(t) + \tan(t)|_{t=0}^{t=\pi/4} = 5\ln|\sqrt{2}+1| - 5\ln|1| = 5\ln(\sqrt{2}+1) \approx 4.4069$. Notes: The identity $\tan(t)^2 + 1 = \sec(t)^2$ is just Pythagoras's $\sin(t)^2 + \cos(t)^2 = 1$, divided by $\cos(t)^2$. $\tan(\pi/4) = 1$ since 45-45-90 triangle with legs 1 and 1, hypotenuse $\sqrt{2}$; for same reason $\sec(\pi/4) = \sqrt{2}$. And $\ln(1) = 0$ because $\ln(xy) = \ln(x) + \ln(y)$ forces this if consider $x = 1$.