↑name↑ ↑ID#↑ May 13, 2003.

# Math 127(3) – Spr 2003 – PRACTICE F ANSWERS – Warren D. Smith.

**1-page "crib sheet" allowed. Calculators allowed.** Warning! 3 errors have been found in earlier draft of the answers...

1. Consider $F(x,y) = \sqrt{x^2 - 3y}$. What is the *domain*-set of this function? Sketch that set.
   ANS. domain is $x$, $y$ so that $x^2 \geq 3y$ which is the region below the parabola $y = x^2/3$.

2. And what is its *range* set?
   ANS. The non-negative reals.

3. What is the level set $F = 4$. Sketch it.
   ANS. It is the parabola $x^2 - 16 = 3y$ which includes as its lowest point $(0, -16/3)$.

4. What is the gradient $\vec{\nabla} F$ of $F$?
   ANS. $\frac{(x,\ -3/2)}{\sqrt{x^2-3y}}$.

5. What is the directional derivative of $F$ at $(x,y) = (2,1)$ in the direction $(1,1)/\sqrt{2}$?
   ANS. $(x - 3/2)/\sqrt{2x^2 - 6y} = (1/2)/\sqrt{8-6} = (1/2)/\sqrt{2} = 1/\sqrt{8}$.

6. What is the angle between the gradient-vector of $F$ at the point $(x,y) = (-5,3)$, and the level set $F = 4$?
   ANS. perpendicular, $90°$, because gradients (direction of fastest increase) are always perpendicular to level sets (sets where value stays at same level)!

7. A parallelogram in 3-dimensional space has 3 vertices $(0,4,7)$, $(1,1,2)$ and $(1,0,-1)$. What is its 4th vertex?
   ANS. Question should have said that those 3 vertices were adjacent *in that order* (otherwise you can't solve it unambiguously). Then the fact that $\vec{A} + \vec{C} = \vec{B} + \vec{D}$ for a parallelogram ABCD in that cyclic order (2 times the parallelogram's center, equals 2 times its center!), so $\vec{D} = \vec{A} + \vec{C} - \vec{B}$ tells us that $(0+1-1, 4+0-1, 7-1-2) = (0,3,4)$ works.

8. (Same parallelogram) What is the angle at that 4th vertex?
   ANS. The edge vectors are $\vec{a} = (-1,3,5)$ and $\vec{b} = (0,-1,-3)$. Their lengths are $|\vec{a}| = \sqrt{35}$ and $|\vec{b}| = \sqrt{10}$ and their dot product is $\vec{a} \cdot \vec{b} = -18$. Angle is $\theta = \arccos \frac{-18}{\sqrt{350}} = \arccos \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|\,|\vec{b}|}$

9. (Same parallelogram) What is its area?
   ANS. The cross product of the edge vectors is $\vec{a} \times \vec{b} = (-1,3,5) \times (0,-1,-3) = (-4,-3,1)$. The area is the length of this vector which is $\sqrt{4^2 + 3^2 + 1^2} = \sqrt{26}$.

10. For $H(x,y) = x^3 + 2xy - 2y^2$, find all local minima, local maxima, and saddle points (and say which are which).
    ANS. The gradient is $\vec{\nabla} H = (3x^2 + 2y, 2x - 4y)$. So any flat spots must obey $3x^2 + 2y = 0$ and $2x - 4y = 0$. Solve... Hence $x = 2y$ and $12y^2 + 2y = 2(6y+1)y = 0$ so that $y = 0$ or $y = -1/6$. So the only flat spots are $(0,0)$ and $(-1/3, -1/6)$. The former is a saddle just by considering the behavior on the line $y = 0$, namely $H = x^3$, which is neither a min nor a max at $x = 0$.

    To handle the latter use the discriminant test. We have $A = H_{xx} = 6x = -2$, $C = H_{yy} = -4$, and $B = H_{xy} = 2$. Hence the discriminant is $D = B^2 - AC = 4 + 8 = 12$. Since $D$ is *negative* this is a *saddle.*

    **WARNING!** I did this wrong on earlier versions of the answers to this practice final!! The test is: If $D > 0$ saddle, if $D < 0$ min(if $a > 0$) or max(if $a < 0$).

11. Maximize $F(x, y, z) = 2x + 3y - 5z$ subject to the constraint $z = x^2 + y^2$. [Hint: Lagrange multiplier trick may help.]
ANS. The gradient $\vec{\nabla} F = (2, 3, -5)$ has to be *parallel* to the gradient $(2x, 2y, -1)$ of the constraint function $x^2 + y^2 - z$. (Lagrange's whole trick.) Hence $x = k$, $y = 3k/2$, and $-1 = -5k$, for some value of the "Lagrange multiplier" $k$. So $k = 1/5$. (**WARNING**! In earlier draft of solutions I had the wrogn equation $z = 5k$ l;eading to nuttiness.) Hence $x = 1/5$ and $y = 3/10$ and hence $z = x^2 + y^2 = 13/100$. Hence $F = 2/5 + 9/10 - 75/100 = 11/20$.

12. A parallelipiped has these edge-vectors emanating from one of its vertices: $(3, -1, 4)$, $(-1, 2, 5)$, and $(1, 7, -1)$. What is its volume?
ANS. It is the determinant of

```
 3 -1  4
-1  2  5
 1  7 -1
```

which is

```
-3*2*1 -1*5*1   - 4*1*7
-3*5*7 +1*1*1   - 4*2*1
```

which is $-151$. Well... of course we don't want the *signed* volume we want the volume so we need to say $+151$.

13. For each of the following decide if it is a vector, a scalar, or senseless:
$(\vec{a} \times \vec{b}) \times \vec{c}$. VECTOR
$(\vec{a} \cdot \vec{b}) \times \vec{c}$. INSANE
$(\vec{a} \times \vec{b}) \cdot \vec{c}$. SCALAR
$(\vec{a} - \vec{b}) \cdot \vec{c}$. SCALAR
$(\vec{a} - \vec{b}) \times \vec{c}$. VECTOR
$(\vec{a} \cdot \vec{b}) - \vec{c}$. INSANE
$(\vec{a} \cdot \vec{b}) + 3$. SCALAR
$(\vec{a} \cdot \vec{b}) + \vec{c}$. INSANE
$(\vec{a} \times \vec{b}) + 3\vec{c}$. VECTOR

14. Here is a curve: $\vec{r}(t) = (3t,\, 9 + t^2)$. Find all points $(x, y) = \vec{r}(t)$ on this curve such that $\vec{r}(t)$ and $\vec{r}'(t)$ are (i) perpendicular (ii) parallel in same direction (iii) parallel in opposite directions.
ANS. Differentiating, we have $\vec{r}'(t) = (3, 2t)$.
(i) The condition for perpendicularity is $(3, 2t) \cdot (3t, 9 + t^2) = 0$ which means that $9t + 18t + 2t^3 = 0$ so that $t = 0$ or $27 + 2t^2 = 0$ (the latter has no real solutions.) The point is then $\vec{r}(0) = (0, 9)$. Then $\vec{r}'(0) = (3, 0)$.
(ii) The condition for parallel or antiparallelism is $(3, 2t)k = (3t, 9 + t^2)$, which means that $t = k$ and $2k^2 = 9 + k^2$ so $k^2 = t^2 = 9$. Use $t = 3$
(since $t = k > 0$ this is parallel in same direction). The point is then $\vec{r}(3) = (9, 18)$. Then $\vec{r}'(3) = (3, 6)$.
(iii) Use $t = -3$. The point is then $\vec{r}(-3) = (-9, 18)$. Then $\vec{r}'(3) = (3, -6)$.

15. Parameterize the curve $5x^2 + 7y^2 = 54$ with a single, everywhere-smooth function of $t$, going clockwise as $t$ increases.
ANS. $x = (\sin t)\sqrt{54/5}$, $y = (\cos t)\sqrt{54/7}$ for $0 \le t < 2\pi$. Note: $y = \pm\sqrt{54 - x^2}$ would not have been have a *single* smooth function. My use of sin for $x$ and cos for $y$ here causes clockwiseness.

16. Write an integral giving the length of that curve (but do not evaluate the integral).
ANS. $\sqrt{54} \int_0^{2\pi} \sqrt{(\cos t)^2/5 + (\sin t)^2/7}\, dt$.

17. Simplify $\vec{a} \times (\vec{b}+\vec{c}) + \vec{b} \times (\vec{a}+\vec{c}) + \vec{c} \times \vec{b}$?
ANS. $\vec{a} \times \vec{c}$. Use distributive law and the anticommutativity of $\times$, then cancel like mad.

18. What is the curl of $(x^2, x+y+z, 5)$?
ANS. $(-1, 0, 1)$.

19. What is the div of $(x^2, x+y+z, 5)$?
ANS. $2x+1+0$.

20. Here is a double integral. $\int_0^9 \int_{\sqrt{y}}^3 \cos(x^3)\mathrm{d}x\mathrm{d}y$. Evaluate it completely. [Hint. What happens if you interchange the order of integration? What are the right bounds on the integrals in that case? May help to sketch the region of integration.]
ANS. $\int_0^3 \int_0^{x^2} \cos(x^3)\mathrm{d}y\mathrm{d}x = \int_0^3 x^2\cos(x^3)\mathrm{d}x = [\sin(x^3)/3]_0^3 = \sin(27)/3$. (**WARNING!** I stupidly wrote this with a cos not a sin in an earlier draft of these answers!!)

21. What function, if any, has gradient $(5x,\ 2y,\ 7+\cos z)$?
ANS. $5x^2/2 + y^2 + 7z + \sin z$.
Actually, that was too easy. OK, try this one instead: What function, if any, has gradient $(5x+7y^2,\ 2y+14xy,\ 7+\cos z)$?
ANS. The magic function $F$ must have $x$-partial$=5x+7y^2$ so $F = 5x^2/2 + 7y^2x + C_1(y,z)$. It also must have $y$-partial$=2y+14xy$ so $F = y^2 + 7xy^2 + C_2(x,z)$. It also must have $z$-partial$=7+\cos z$ so $F = 7z + \sin z + C_3(x,y)$. Putting these all together, we find $F(x,y,z) = 5x^2/2 + 7y^2x + y^2 + 7z + \sin z + C$ works.

22. What is the curve integral $\int (5x, 2y, \cos(z)+7) \cdot \mathrm{d}\vec{s}$ along *any* curve from $(-1,4,5)$ to $(2,0,0)$?
ANS. Use Newton theorem $\int_{\text{curve}} (\vec{\nabla}F) \cdot \mathrm{d}\vec{s} = F(\text{end}) - F(\text{start})$. Ans is $(5/2 + 4^2 + 35 + \sin 5) - (10) = (43.5 + \sin 5)$.

23. What is $\int\int\int (7+x+xy+x^2)\mathrm{d}x\mathrm{d}y\mathrm{d}z$ over the $2\times4\times6$ brick $|x|<1$, $|y|<2$, $|z|<3$?
ANS. The $x$ and $xy$ terms cancel out to 0 on integration due to symmetry. The 7 leads to 7volume $= 7\times2\times4\times6 = 336$. The $x^2$ integrates $\mathrm{d}x$ to $2/3$, then this is just a constant as far as $y$ and $z$ are concerned hence leads to $4\times6\times2/3 = 16$. Total is $16+336 = 352$.

24. What is the flux of the smooth vector field $\vec{F}(x,y,z) = (x, y, z+\arctan(5+xy))$ out through the surface of that brick?
ANS. $\vec{\nabla}\cdot\vec{F} = 1+1+1 = 3$. So the divergence theorem tells us the answer is $\int\int\int 3\mathrm{d}x\mathrm{d}y\mathrm{d}z = 3\times$ volume $= 3\times2\times4\times6 = 144$.

25. Let $\vec{H}(x,y) = (xy+3+\cos x,\ xy+\sinh y)$. What is the circulation integral $\int \vec{H}\cdot\mathrm{d}\vec{s}$ anticlockwise around the closed curve bounding the region $3x^2 < y < 12$ in the $xy$ plane. [Hint: Green or Stokes.]
ANS. The curl of $\vec{H}$ is $\vec{\nabla}\times\vec{H} = (0,\ 0,\ y-x)$. So upon using Stokes theorem (or more simply, Green theorem, i.e. the 2D flat version of Stokes where only the $z$-component of the curl matters) we get
$\int \vec{H}\cdot\mathrm{d}\vec{s} = \int_{-2}^{2}\int_{3x^2}^{1} 2(y-x)\mathrm{d}y\mathrm{d}x$
since the $x$ cancels out to 0 by symmetry for $+x$ and $-x$, we only need to keep the $y$:
$= \int_{-2}^{2}\int_{3x^2}^{1} 2y\mathrm{d}y\mathrm{d}x = \int_{-2}^{2}[y^2/2]_{3x^2}^{1} 2\mathrm{d}x = \int_{-2}^{2}(72-9x^4/2)\mathrm{d}x = [72x - 9x^5/10]_{-2}^{+2}\mathrm{d}x = 1152/5$.

26. What is the vector element of surface area of the parameterized surface $(x,y,z) = (\sin p,\ \cos q,\ p+q)$?
ANS. $\mathrm{d}\vec{\text{area}} = \frac{\partial\vec{r}}{\partial p}\times\frac{\partial\vec{r}}{\partial q}\mathrm{d}p\mathrm{d}q = (\cos p,\ 0,\ 1)\times(0,\ -\sin q,\ 1)\mathrm{d}p\mathrm{d}q = (\sin q,\ -\cos p,\ -\cos p\sin q)\mathrm{d}p\mathrm{d}q$.

27. What is the surface area (over the parameter region $0<p<\pi/2$, $0<q<\pi/2$)?
ANS. The scalar area element is the length of the vector area element:
darea$= |\mathrm{d}\vec{\text{area}}| = \sqrt{\sin^2 q + \cos^2 p + \cos^2 p \sin^2 q}\,\mathrm{d}p\mathrm{d}q$ so the surface area is
area$= \int_0^{\pi/2}\int_0^{\pi/2}\sqrt{\sin^2 q + \cos^2 p + \cos^2 p\sin^2 q}\,\mathrm{d}p\mathrm{d}q$.
How do we possibly do this integral? Well, uh, sorry about that, but doing this integral is essentially impossible so we have to leave it unevaluated. (In the real final, I'll give you integrals you can do, or instruct you not to bother trying.)