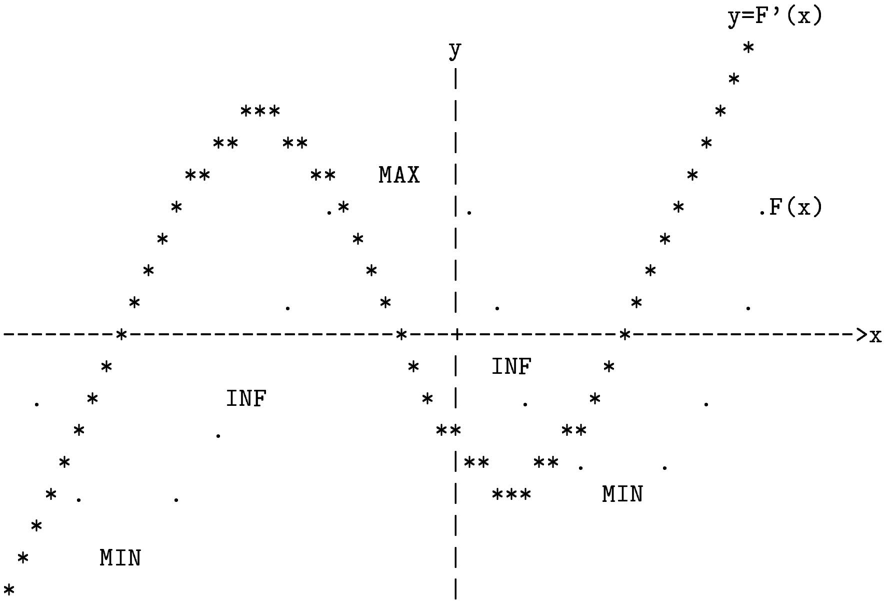
↑name↑ ↑ID#↑ December 9, 2002. Show all work.

Math 75 – Fall 2002 – Hmwk #10 ANSWERS – Warren D. Smith

This homework is probably a little harder than what the real final exam (2 hours long) will be like. Feel free to use extra sheets, the real final would have a lot more white space.



1. Find derivative (d/dx) of these: $\sqrt{\cos(x)} + 8x^{2/7}$

   ANS: $-(1/2)(\cos x)^{-1/2}\sin(x) + (16/7)x^{-5/7}$.

2. $\ln\left(\frac{\sqrt{5x+7}}{8x+1}\right)$ [Hint. It might be a lot easier if you first use properties of logs to rewrite the problem in a different, but equivalent, way, *then* solve the problem.]

   ANS: First rewrite the problem as that of taking the deriv of $\frac{1}{2}\ln(5x+7) - \ln(8x+1)$. Here we have used $\ln(A/B) = \ln(A) - \ln(B)$ to split into a difference of two logs, then used $\ln\sqrt{x} = \frac{1}{2}\ln x$ to get rid of the square root (this is the special case $p = 1/2$ of the identity $\ln(x^p) = p\ln x$). Now this new problem is easy to take the derivative of. Answer is $\frac{5/2}{5x+7} - \frac{8}{8x+1}$ by the chain rule.

3. $\frac{x^2+7}{99x-e^{5x}}$

   ANS: quotient rule. $\frac{(2x)(99x-e^{5x})-(x^2+7)(99-5e^{5x})}{(99x-e^{5x})^2}$

4. $e^{\cos x} - \tan(e^{3x})$

   ANS: several chain rule uses. The last one is $(e^{3x})' = 3e^{3x}$. Answer is $-\sin(x)e^{\cos(x)} - 3\sec(e^{3x})^2 e^{3x}$

5. $x^e + e^x + x^x + e^e$

   ANS: $ex^{e-1} + e^x + x^x(1+\ln x) + 0$. This is a good example of the need to realize which things are *constants*, i.e. stay same when x changes. e is a constant. x is not. The x^x term is handled by rewriting it as $x^x = e^{x\ln x}$. Then, use chain rule to get that its derivative is $e^{x\ln x}(\ln x + x/x)$ which simplifies to $(1+\ln x)e^{x\ln x}$ which simplifies to $(1+\ln x)x^x$.

6. Find the numerical value of dy/dx if $2x^3y - x^2 + y^3 = 11$, at the point $x = 1$, $y = 2$.

   ANS: 1. differentiate both sides: $2x^3y' + 6x^2y - 2x + 3y^2y' = 0$.
   2. regroup terms: $2x^3y' + 3y^2y' = 2x - 6x^2y$. (Note how terms moved to the right hand side of eq now

have opposite signs.)
3. solve for $y' = (2x - 6x^2y)/(2x^3 + 3y^2)$.
4. Evaluate at $x = 1$ and $y = 2$ to get $y' = (2 - 12)/(2 + 12) = -10/14 = -5/7$.
This was easy to make a mistake on, so a *sanity check* would help. Here is how you can do that. Plug in $x = 1.001$ and $y = 2 - (5/7)0.001 = 1.999285714$ to the original equation $2x^3y - x^2 + y^3 = 11$. We get $11.00000977 \approx 11$, GOOD! It almost worked (but not exactly) because the tangent line is not the same as the curve, just close. If we had wrongly thought (as one solver did) that the answer was not $-5/7$ but instead $-5/4$, the sanity check of plugging in $x = 1.001$, $y == 2 - (5/4)0.001 = 1.998750000$ would have yielded $10.99251287 = 11$, which is not nearly as good agreement. Generally we should expect agreement to about twice as many decimal places as the perturbation. Here the perturbation was in the 3rd decimal so we hope for agreement to about 6 decimals. Sure enough, the sane sanity check was off in the 6th decimal place, but the insane one was off in the 3rd place.

7. What is the equation of the tangent line (also called: linear approximation) to the curve $y = x^{1/3} + 5x^2 + \ln(3x^2 + 1)$ at $x = 1$? [You would not be given this **hint** on the real final, but: a method is (1) find the slope s of this curve at $x = 1$ by finding an equation for y' then substituting $x = 1$ into it, (2) find the value v of y at $x = 1$, (3) use the fact that the equation of a line with slope s passing through a point $(x = B, y = v)$ is $y_{\text{line}} = (x - B)s + v$, (4) write down this line-equation in our case! (5) perhaps try a sanity check.]

   ANS: 1. Start by finding $y'(x) = (1/3)x^{-2/3} + 10x + \frac{6x}{3x^2+1}$ (power rules and chain rule).
   2. Evaluate $s = y'(1) = (1/3) + 10 + 6/4 = 71/6$.
   3. Evaluate $v = y(1) = 1 + 5 + \ln(4) = 6 + \ln(4)$.
   4. Write answer $y_{\text{line}} = (x - 1)(71/6) + [6 + \ln(4)]$.

8. If the distance an object has traveled after t seconds is t^3 meters, then what is the object's speed when $t = 5$?

   ANS: Let $f(t) = t^3$. Then the speed is $f'(t) = 3t^2$ at time t, and when $t = 5$ this is $f'(5) = 75$ meters/sec.

9. And what is its acceleration?

   ANS: The acceleration is $f''(t) = 6t$ at time t, and when $t = 5$ this is $f''(5) = 30$ meters/sec^2.

10. Find these limits (Answer could be "$+\infty$" or "$-\infty$" or "does not exist" incidentally.) [Show all work. Trying to estimate a numerical value with a calculator is a fine and recommended sanity check, but they will want you to show your work and methods and will give such an approach a lot less credit than if you get the *exact* answer via an *exact* method.] $\lim_{x\to 3+} \frac{14}{x-3}$

    ANS: $+\infty$. (As 2-sided limit $\lim_{x\to 3+} \frac{14}{x-3}$ ans would have been "does not exist.")

11. $\lim_{x\to 1} \frac{x^2-6x+5}{(x-1)^2}$

    ANS: Does not exist. One way to handle this is to notice that $x^2 - 6x + 5 = (x - 1)(x - 5)$ so the expression simplifies (after canceling an $(x - 1)$ to $(x - 5)/(x - 1)$, which when $x \to 1$ becomes $-4/0$ which goes to $\pm\infty$ depending from which side x approaches 1. Since these *differ* the limit does not exist.

    A different way is to use L'Hopital (since the original expression gives $0/0$). We get $\lim_{x\to 1} \frac{2x-6}{2(x-1)}$ which yields $-4/0$ and now the same reasoning says $\pm\infty$ so D.N.E.

12. $\lim_{h\to 0} \frac{\cos(x+h)-\cos(x-h)}{2h}$

    ANS: $\sin(x)$. This is a modification of the usual definition of the derivative of $\cos(x)$. Notice that this is still a rise over a run in the limit when the run gets very small, it is just *centered* at x rather than off to one side (i.e. going from $x - h$ to $x + h$ rather than from x to $x + h$). So you still get the slope, i.e. derivative, of $\cos(x)$ in the limit $h \to 0$.

    Another way to do this is to use L'Hopital.

13. $\lim_{x\to 0} \frac{|x|}{x}$

ANS: Does not exist! This is a trick question of the sort they love to put on finals. Note if $x > 0$ then $|x| = x$ so $|x|/x = 1$. But if $x < 0$ then $|x| = -x$ (which is positive) so $|x|/x = -1$ by cancelling x's. So the limit is $+1$ or -1 depending which side $x \to 0$ from, so the 2-sided limit doesn't exist since these 1-sided limits *differ*.

14. *I* have plotted $F'(x)$. *Your* job is to plot $F(x)$, given that $F(0) = 0$. On your plot of $F(x)$, please indicate (and label as such) all its local (or global) mins, local (or global) maxes, and inflection points.

ANS: See plot at start, connect dots for F(x)

15. A triangle has legs of lengths 10 and 12 meters, and the angle between them is T.
(a) Suppose T is increasing at 1 degree per second. What is the rate of increase in the area of the triangle when $T = 60$degrees? [Warning: watch out for confusion about radians versus degrees! You may want to first solve under the assumption T is increasing at 1 *radian* per second, then ask what happens to your answer if you change 1 radian/sec to 1 degree/sec.]
(b) What is the maximum area the triangle will ever have?

ANS(a): The area is $A(T) = 10 \cdot 12 \cdot \sin(T)/2 = 60\sin(T)$ meters^2. (Book frontspiece, area of triangle with legs a, b enclosing angle θ is $= (ab\sin\theta)/2$.) So $A'(T) = 60\cos(T)$ would be the rate of increase of area, if T were increasing 1 radian/sec. But in fact T is increasing a 1 *degree* per sec, which is $\pi/180$ times faster. Hence the rate of increase of area really is $(\pi/180)\cdot 60\cos(T)$. Since cos(60 degrees) $= 1/2$, answer is rate of increase is $(\pi/180)(60/2) = \pi/6$ square-meters per sec.

ANS(b): $A'(T) = 60\cos(T)$ is 0 (critical point) when $T = \pi/2$ (i.e. $T = 90$ degrees). Then $\sin(T) = 1$. So, $A_{\max} = 60$, achieved when the triangle becomes a *right* triangle. We now see right triangles always have the largest possible area among all triangles with their leg lengths, since height is maximized.

16. Cobalt-60 is a radioactive isotope with a half life of 5.25 years. That is, half the Cobalt-60 atoms present in a brick at any moment will not be there 5.25 years later. How many years until only 1/10000 of the original stock of Cobalt-60 atoms remain? Express your answer as both an exact formula and as a number.

ANS: $5.25\ln(1/10000)/\ln(1/2) \approx 69.76$ years. [Why: After $5.25n$ years $(1/2)^n$ of the cobalt is left. We want $(1/2)^n = 1/10000$. Take ln of both sides to get $n\ln(1/2) = \ln[(1/2)^n] = \ln[1/10000]$. Solve for n to get $n = \ln(1/10000)/\ln(1/2)$.]

17. State TWO DIFFERENT functions $F(x)$ such that $F'(x) = 8x^2 + 17$.

ANS: $8x^3/3 + 17x + 1$ and $8x^3/3 + 17x + 0$. Indeed $8x^3/3 + 17x + C$ for any constant C.

18. If $F(x) = |x^2 - 9|$,
(a) what is the global min of $F(x)$ [state both the x and the $F(x)$ value] for $-4 \le x \le 4$?
(b) What is the global max of $F(x)$ [state both the x and the $F(x)$ value] for $-4 \le x \le 4$? [Hint: watch out for the effects of the absolute value signs!]

ANS: If $|x| < 3$ then $x^2 - 9$ is negative so $|x^2 - 9| = 9 - x^2$. Otherwise, $|x^2 - 9| = x^2 - 9$. There are two co-equal global **mins** when $x = \pm 3$, there get $F(x) = 0$. This is minimum possible since due to the absolute value signs, $F(x) \ge 0$ always.

This question was intended to be an example of a case where the min is one of those critical points at which the derivative of $F(x)$ *does not exist.* (The graph has a "corner" here.) The critical points here are $x = \pm 3$ ($F'(x)$ does not exist), $x = \pm 4$ (interval endpoints) and $x = 0$ ($F'(x) = -2x = 0$ here).

Now, examining these critical points we see $F(\pm 4) = |16 - 9| = 7$ and $F(\pm 3) = 9 - 9 = 0$ and $F(0) = |0 - 9| = 9$. So the global **max** is at $x = 0$, $F(0) = 9$ since $9 > 7$.