↑name↑ ↑ID#↑ November 19, 2002. Show all work.

Math 75 – Fall 2002 – Practice quiz #3 – Warren D. Smith

1. Suppose $F(y)$ is the inverse function of $G(x) = x^3 + x + 1$ (defined when $x < 0$). Write a formula, solely in terms of y and $F(y)$, for $F'(y)$.

   ANSWER: $F(G(x)) = x$ since F, G inverse fns. Derivative is $F'(G(x))G'(x) = 1$. That is $F'(G(x)) = 1/G'(x)$. Now let $y = G(x)$ (renaming) and then $x = F(y)$ to get $F'(y) = 1/G'(F(y))$. Could just remember this formula for derivative of inverse fn and start from here instead. Since $G'(x) = 3x^2 + 1$ for us, $F'(y) = 1/[3F(y)^2 + 1]$ is answer.

2. Suppose x and y are related by $x^3 \cos(y) - \sin(xy) = 3x$. Write a formula, in terms of y and x, for $y'(x)$.

   ANSWER: Implicit differentiation [first term by product rule, 2nd by chain rule] to get
   $3x^2 \cos(y) - x^3 \sin(y)y' - \cos(xy)(xy' + y) = 3$, rewrite with coefficients of y' and not of y' grouped into two groups: $3x^2 \cos(y) - \cos(xy)y - 3 = +x^3 \sin(y)y' + \cos(xy)xy'$,
   then solve for y' to get answer $y' = [3x^2 \cos(y) - \cos(xy)y - 3]/[x^3 \sin(y) + \cos(xy)x]$.

3. The voltage $V(t)$ coming out of a wall socket at time t is $V(t) = 170 \sin(120\pi t)$ where t is in seconds.
   (a) What is the maximum voltage ever achieved (at the worst possible time to stick your finger in the socket)?
   (b) The "average height" of a curve $y = F(x)$ in the region $a < x < b$ is the area under that curve divided by $b - a$. What is the area under the curve $y = V(t)$, in the t, y plane, when $0 < t < 1/120$? (c) What is the average voltage during the timespan $0 < t < 1/120$? (d) What about during the timespan $0 < t < 1/60$?

   ANSWERS: (a) The max is when the sin is 1 (largest $\sin(x)$ can ever be). I.e. when it is 170 volts.
   (b) $\int_0^{1/60} 170 \sin(120\pi t)\mathrm{d}t$. Put this down even if you cannot do the integral, that way you get partial credit. To do the integral, we need a magic function $F(t)$ such that $F'(t) = 170 \sin(120\pi t)$. Then the answer will be $F(1/60) - F(0)$. Well, we know cos's and sin's are friends so try as a guess $F(t) = Q \cos(Rt)$. We do not know what Q and R are yet, the idea is to find that out later: we differentiate using chain rule to get $F'(t) = -QR \sin(Rt)$, then we see $-QR = 170$ and $R = 120\pi$. That means $Q = -170/(120\pi) = -17/(12\pi)$. So great, we know that $F(t) = -17/(12\pi) \cos(120\pi t)$. So the answer is area$= F(1/60) - F(0) = 17/(6\pi)$. We have used $\cos(0) = 1$ and $\cos(\pi) = -1$.
   (c) The average voltage is the area divided by $1/120$ which is $120 \cdot 17/(6\pi) = 340/\pi \approx 108.2$ volts.
   (d) The average voltage over this twice as large timespan is the average over a full cycle (when its argument goes from 0 to 2π) of the sin function and hence averages out to 0.

4. What is the derivative (d/dx) of the following functions of x:
   (a) $\arcsin(x^2) \sin(x)$
   (b) $x 2^{\sin(x)}$
   (c) $\sin(x)^{99}$
   (d) $\arctan(x^3)$

   ANSWERS: (a) $2x \frac{\sin(x)}{\sqrt{1-x^4}} + \arcsin(x^2) \cos(x)$ [have used produict rule and the fact $\arcsin'(x) = 1/\sqrt{1-x^2}$ and the chain rule with x^2 being the inner function.]
   (b) $2^{\sin(x)} + x 2^{\sin(x)} \cos(x) \ln(2)$. [have used product rule. Also have used $2^t = e^{t \ln 2}$ since $A^B = e^{B \ln A} = (e^{\ln A})^B$, which makes it easy to differentiate since now rewritten in terms of known functions ln and exp. Also used chain rule to differentiate $2^{\sin(x) \ln(2)}$ with $\sin(x) \ln 2$ as the inner function of x.]
   (c) $99 \sin(x)^{98} \cos(x)$ by power rule, and chain rule with inner function $\sin(x)$.
   (d) $\frac{3x^2}{1+x^6}$. Used $\arctan'(t) = 1/(1 + t^2)$, and chain rule with inner function$(x) = x^3$, whose derivative is $3x^2$.

5. Find the Taylor series (only necessary to include terms up to and including x^3) of $\ln(1 + e^x)$ based at $x = 0$.

ANSWER: $\ln(1+e^x) = \ln(2) + \frac{1}{2}x + \frac{1}{8}x^2 + 0x^3 + \ldots$ This was got from the Taylor series master formula $F(x) = F(0) + F'(0)x + F''(0)x^2/2 + F'''(0)x^3/3! + F''''(0)x^4/4! + \ldots$ by using these ingredients:
$F(x) = \ln(1+e^x)$, $F(0) = \ln(2)$;
$F'(x) = e^x/(1+e^x)$ [from chain rule], $F'(0) = \frac{1}{2}$;
$F''(x) = e^x/(1+e^x)^2$ [from quotient rule], $F''(0) = \frac{1}{4}$;
$F'''(x) = \frac{e^x}{(1+e^x)^2} - 2\frac{(e^x)^2}{(1+e^x)^3}$, $F'''(0) = 0$.
Sanity check: try the Taylor series at $x = 0.1$ versus the real thing: $\ln(2) + \frac{1}{2}0.1 + \frac{1}{8}0.01 + 0 = 0.744397$; $\ln(1+e^{0.1}) = 0.744397$.

6. Minimize $y = \sqrt{x^3 - x + 1}$ when $x \geq 0$. What is x and what is y at the minimum?

   ANSWER: It is easier to minimize y^2 than y. $(y^2) = x^3 - x - 1$. Take deriv: $(y^2)' = 3x^2 - 1$. Set deriv$= 0$ to find critical point: $3x^2 - 1 = 0$ so $x = \pm\sqrt{1/3}$ and since we have demanded $x \geq 0$, $x = +\sqrt{1/3} = 3^{-1/2} \approx 0.5773502692$. Now $y^2 = 3^{-3/2} - 3^{-1/2} + 1 = 1 - (2/9)\sqrt{3} \approx 0.6150998205$. So $y = \sqrt{1 - (2/9)\sqrt{3}} \approx 0.7842829977$. Sanity check: compute y at $x + 0.01$ and at $x - 0.01$, results are, e.g., 0.7865 which *are* larger so this *is* a min.

7. For which real x is $x^3 - x + 1$ concave-∪?

   ANSWER: The derivative is $3x^2 - 1$ and the 2nd derivative is $6x$. We are concave-∪ when the 2nd derivative is *positive* i.e. when $x > 0$.

8. What is 9! (the factorial of 9)?

   ANSWER: $9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 362880$.

   NOTE: this is probably harder than the real quiz will be, and it incorporates a lot of useful labor-saving tricks. The real quiz will permit a 1-page long crib sheet.