Perron-Frobenius Properties of General Matrices

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PERRON-FROBENIUS PROPERTIES OF GENERAL MATRICES*

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Dedicated to Hans Schneider on the occasion of his 80^{th} birthday

Abstract. A matrix is said to have the Perron-Frobenius property if it has a positive dominant eigenvalue that corresponds to a nonnegative eigenvector. Matrices having this and similar properties are studied in this paper. Characterizations of collections of such matrices are given in terms of the spectral projector. Some combinatorial, spectral, and topological properties of such matrices are presented, and the similarity transformations preserving the Perron-Frobenius property are completely described. In addition, certain results associated with nonnegative matrices are extended to matrices having the Perron-Frobenius property.

Key words. Perron-Frobenius property, generalization of positive matrices, eventually nonnegative matrices, eventually positive matrices.

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1. Introduction.

We say that a real or complex matrix A is nonnegative (positive, nonpositive, negative, respectively) if it is entry-wise nonnegative (positive, nonpositive, negative, respectively) and we write $A \geq 0$ ($A > 0$, $A \leq 0$, $A < 0$, respectively). This notation and nomenclature is also used for vectors.

In 1907, Perron [24] proved that a square positive matrix has the following properties:

1. Its spectral radius is a simple and a positive eigenvalue.
2. The eigenvector corresponding to the spectral radius can be chosen to be (entry-wise) positive (called a Perron vector).
3. No other eigenvalue has a positive eigenvector.
4. The spectral radius is a strictly increasing function of the matrix entries.

Later in 1912, this result was extended by Frobenius [10] to nonnegative irreducible matrices and consequently to nonnegative matrices, using a perturbation argument. In the latter case, there exists a nonnegative dominant eigenvalue with a corresponding nonnegative eigenvector.

These results, known now as the Perron-Frobenius theory, have been widely applied to problems with nonnegative matrices, and also with M-matrices and H-matrices; see, e.g., the monographs [2], [18], [28], [34]. Applications include stochastic processes [28], Markov chains [32], population models [22], solution of partial differential equations [1], and asynchronous parallel iterative methods [11], among others.

A natural question is: which matrices other than nonnegative ones have some of the properties 1–4? Furthermore, which results associated with nonnegative matrices can be extended to other classes of matrices satisfying these properties?

Eventually nonnegative and eventually positive matrices do satisfy some of the properties 1–4. A matrix A is said to be *eventually nonnegative* if $A^k \geq 0$ for all $k \geq k_0$ for some positive integer k_0. Similarly, A is said to be *eventually positive* if $A^k > 0$ for all $k \geq k_0$ for some positive integer k_0.
Friedland [9] was apparently the first to study eventually nonnegative matrices, and

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showed that for such matrices the spectral radius is an eigenvalue. Perron-Frobenius type properties exhibited by eventually positive and eventually nonnegative matrices were further studied in [6], [14], [15], [16], [23], [29], [30], [35], [36]; see also [17]. In particular, Carnochan Naqvi and McDonald [6] studied combinatorial properties of eventually nonnegative matrices whose index is 0 or 1 by considering their Frobenius normal forms, and Eschenbach and Johnson [8] gave combinatorial characterization of matrices that have their spectral radius as an eigenvalue.

In a series of papers [20], [21], [33], Tarazaga and his co-authors extended the Perron-Frobenius theory to matrices with some negative entries and studied closed cones of matrices whose central ray is ee^T, the matrix of ones, giving the maximal angles for which eigenvalue dominance and eigenvector positivity are retained. In [21], limitations of extending the Perron-Frobenius theory outside the cone of positive matrices are discussed.

We mention in passing the work of Rump [26], [27], who generalized the concept of a positive dominant eigenvalue, but this is not related to the questions addressed in this paper.

We call a column or a row vector v *semipositive* if v is nonzero and nonnegative. Likewise, if v is nonzero and nonpositive then we call v *seminegative*. We denote the spectral radius of a matrix A by $\rho(A)$. Following [23], we say that a real matrix A possesses the *Perron-Frobenius property* if A has a positive dominant eigenvalue with a corresponding nonnegative eigenvector. We say that A possesses the *strong Perron-Frobenius property* if A has a simple, positive, and strictly dominant eigenvalue with a positive eigenvector. If a matrix A satisfies $Av = \rho(A)v$ for some semipositive vector v, then we say that A has a *Perron-Frobenius Eigenpair* $(\rho(A), v)$. In the latter case, if $\rho(A) > 0$, we call v a *right Perron-Frobenius eigenvector* for A. Similarly, if $\rho(A) > 0$, and $w^T A = \rho(A)w^T$ for some semipositive vector w then we call w a *left Perron-Frobenius eigenvector* for A.

Following [20], we let PFn denote the collection of $n \times n$ real matrices whose spectral radius is a simple, positive, and strictly dominant eigenvalue having positive left and right eigenvectors. Equivalently, we can say that PFn is the collection of matrices A for which both A and A^T possess the strong Perron-Frobenius property. Similarly, WPFn denotes the collection of $n \times n$ real matrices whose spectral radius is a positive eigenvalue having nonnegative left and right eigenvectors. Equivalently, WPFn is the collection of matrices A for which both A and A^T possess the Perron-Frobenius property.

In this paper, we prove some new results for matrices in these sets. These new results are the counterparts to those known for nonnegative matrices (sections 3 and 4). We further study these sets of matrices from several points of view. We first show how they relate to one another (section 5), and how the sets are invariant when the matrices are raised to powers (section 6). One of the questions we answer is: for which similarity transformations are these sets invariant? (section 7). We give new characterizations of PFn and of WPFn in terms of spectral projectors (section 8). We study some topological aspects of theses sets: In section 9, we show that the set of symmetric matrices in PFn extends beyond a known cone centered at the matrix of all ones. In section 10, we study the connectedness of the collection of matrices having the Perron-Frobenius property and other subcollections. We also show some application of our results to the analysis of the signs of singular vectors of matrices in these sets (section 11).

2. Further notation and preliminary definitions. The spectrum of matrix A is denoted by $\sigma(A)$. We call an eigenvalue of A a simple eigenvalue if its algebraic multiplicity in the characteristic polynomial is 1. We call an eigenvalue $\lambda \in \sigma(A)$ *dominant* if $|\lambda| = \rho(A)$. We call an eigenvalue $\lambda \in \sigma(A)$ *strictly dominant* if $|\lambda| > |\mu|$ for all $\mu \in \sigma(A)\backslash\{\lambda\}$. The algebraic multiplicity of an eigenvalue $\lambda \in \sigma(A)$ is its multiplicity as a root of the characteristic polynomial and is denoted by $mult_\lambda(A)$, while the index of an eigenvalue $\lambda \in \sigma(A)$ is its multiplicity as a root of the minimal polynomial of A and is denoted by $index_\lambda(A)$. Sometimes, as a shorthand, we write *index of* A for $index_0(A)$. The ordinary eigenspace of A for the eigenvalue λ is denoted by $E_\lambda(A)$. By definition, $E_\lambda(A) = \mathcal{N}(A - \lambda I)$, the null space of $A - \lambda I$. The nonzero vectors in $E_\lambda(A)$ are called ordinary eigenvectors of A corresponding to λ. The generalized eigenspace of A for the eigenvalue λ is denoted by $G_\lambda(A)$. Note that $G_\lambda(A) = \{v \mid (A - \lambda I)^k v = 0 \text{ where } k = index_\lambda(A)\} = \mathcal{N}(A - \lambda I)^k$. $G_\lambda(A)$ is also known as the algebraic eigenspace of A for the eigenvalue λ. The nonzero vectors of $G_\lambda(A)$ are called generalized eigenvectors for A corresponding to λ. We call the projection operator onto $G_\lambda(A)$ a spectral projector if $|\lambda| = \rho(A)$ and the projection is along the direct sum of the other generalized eigenspaces. For any complex number λ, $J_s(\lambda)$ denotes an $s \times s$ Jordan block corresponding to λ, i.e., $J_s(\lambda) = \lambda I_s + N_s$ where I_s is the $s \times s$ identity matrix and N_s is the matrix whose first superdiagonal consists of 1's while all other entries are zeroes. Note that $N_s = 0$ if $s = 1$. The $s \times s$ zero matrix is denoted by O_s. When the dimension of the zero matrix is clear we just write O. The Jordan canonical form of matrix A is denoted by $J(A)$. For any eigenvalue $\lambda \in \sigma(A)$, $B(\lambda)$ denotes the Jordan box corresponding to λ in $J(A)$, i.e., $B(\lambda)$ is the direct sum of all of the Jordan blocks corresponding to λ in $J(A)$.

We say that $A \in \mathbb{C}^{1\times 1}$ is reducible if $A = [0]$. We say that $A \in \mathbb{C}^{n\times n}$ $(n \geq 2)$ is reducible if A is permutationally similar to $\begin{bmatrix} B & O \\ C & D \end{bmatrix}$ where B and D are square matrices. We say that a matrix $A \in \mathbb{C}^{n\times n}$ $(n \geq 1)$ is irreducible if A is not reducible.

For an $n \times n$ matrix A, we define the (directed) graph $G(A)$ to be the graph with vertices $1, 2, \ldots, n$ in which there is an edge (i, j) if and only if $a_{ij} \neq 0$. A path from vertex j to vertex m is a sequence of vertices $v_1, v_2, \ldots, v_t$ such that (v_l, v_{l+1}) is an edge in $G(A)$ for $l = 1, \ldots, t$ where $v_1 = v_{t+1} = j$ and $v_t = m$. We say a vertex j has access to a vertex m if there is a path from j to m in $G(A)$. If j has access to m and m has access to j, we say j and m communicate. The communication relation is an equivalence relation on $\{1, 2, \ldots, n\}$ and an equivalence class α is called a class of A. If s is the number of equivalence classes under the communication relation, we define the reduced graph of A, denoted by $R(A)$, as the graph whose vertices are $1, 2, \ldots, s$ in which each vertex represents a class of A and there is an edge (j, m) in $R(A)$ if and only if there is an edge in $G(A)$ from a vertex in the class that j represents to a vertex in the class that m represents. If G is any graph on the set of vertices $1, 2, \ldots, n$, then we define its transitive closure $\overline{G}$ as a graph on the same set of vertices in which (j, m) is an edge in $\overline{G}$ if and only if j has access to m in G.

We call a collection $\{\alpha_1, \alpha_2, \cdots, \alpha_m\}$ of subsets of $\{1, 2, \ldots, n\}$ a partition of $\{1, 2, \ldots, n\}$ if $\cup_{i=1}^m \alpha_i = \{1, 2, \ldots, n\}$ and $\alpha_i \cap \alpha_j = \phi$ whenever $i \neq j$. Moreover, we call the m-tuple $(\alpha_1, \alpha_2, \cdots, \alpha_m)$ an ordered partition of $\{1, 2, \ldots, n\}$. If $A \in \mathbb{C}^{n\times n}$, $v \in \mathbb{C}^n$, and $\alpha, \beta \subset \{1, 2, \ldots, n\}$, then $A[\alpha, \beta]$ denotes the submatrix of A whose rows are indexed by α and whose columns are indexed by β. If $\alpha = \beta$, then we write $A[\alpha]$ for the principal submatrix of A whose rows and columns are indexed by α. Moreover, $v[\alpha]$ is the subvector of v with entries indexed by α. If A is in $\mathbb{C}^{n\times n}$ and

$\kappa = (\alpha_1, \alpha_2, \cdots, \alpha_m)$ is an ordered partition of $\{1, 2, \ldots, n\}$, then A_κ will denote the block matrix whose $(i,j)^{th}$ block is $A[\alpha_i, \alpha_j]$. If $A \in \mathbb{C}^{n\times n}$ and $\alpha \subset \{1, 2, \ldots, n\}$ is a class of A, then we call $G(A[\alpha])$ a strong component of $G(A)$. If $A \in \mathbb{C}^{n\times n}$ and α is a class of A, then we say that α is a basic class if $\rho(A[\alpha]) = \rho(A)$, an initial class if $G(A[\alpha])$ is not accessed by any other strong component in $G(A)$, and a final class if $G(A[\alpha])$ does not access any other strong component in $G(A)$.

3. The classes of a matrix in WPFn. It is well-known that a nonnegative matrix A has positive left and right eigenvectors corresponding to $\rho(A)$ if and only if the classes of A are all basic and they are either all initial or all final; see, e.g., [2, Ch. 2, Section3]. In this section, we study analogous necessary and sufficient conditions on the classes of a matrix so that it is in WPFn. Unlike the nonnegative case, these necessary and sufficient conditions are not the same. They are presented in two complementary theorems; see also the end of section 4 for a special case.

THEOREM 3.1. *Let A be a matrix in WPFn.*

(i) If α is a final class of A and $v[\alpha]$ is nonzero for some right eigenvector v of A corresponding to $\rho(A)$, then α is a basic class.

(ii) If α is an initial class of A and $w[\alpha]$ is nonzero for some left eigenvector w of A corresponding to $\rho(A)$, then α is a basic class.

Proof. In general, for any class α of A, we have

$$(Av)[\alpha] = A[\alpha]v[\alpha] + \sum_{\beta} A[\alpha, \beta]v[\beta],$$

where the sum on the right side is taken over all classes β that are accessed from α but are different from α. When α is a final class and v is an eigenvector of A corresponding to $\rho(A)$, we have

$$\rho(A)v[\alpha] = (Av)[\alpha] = A[\alpha, \alpha]v[\alpha].$$

If, in addition, $v[\alpha]$ is nonzero, we can conclude that α is a basic class. This proves (i). The proof of (ii) is analogous. □

THEOREM 3.2. *If $A \in \mathbb{R}^{n\times n}$ has two classes α and β, not necessarily distinct, such that:*

α is basic, initial, and $A[\alpha]$ has a right Perron-Frobenius eigenvector
β is basic, final, and $A[\beta]$ has a left Perron-Frobenius eigenvector,

then $A \in$ WPFn.

Proof. There is a semipositive vector v such that $A[\alpha]v = \rho(A)v$. Define the vector $\tilde{v} \in \mathbb{R}^n$ as follows: for any class γ of A, $\tilde{v}[\gamma] = v$ if $\gamma = \alpha$, and $\tilde{v}[\gamma] = 0$ if $\gamma \neq \alpha$. It is easily seen that $\tilde{v}$ is semipositive and that $A\tilde{v} = \rho(A)\tilde{v}$. Similarly, using the class β, there is a semipositive vector $\tilde{w}$ for which $\tilde{w}^T A = \rho(A)\tilde{w}^T$. Hence, $A \in$ WPFn. □

4. The classes of an eventually nonnegative matrix and its algebraic eigenspace. Carnochan Naqvi and McDonald [6] showed that if A is eventually nonnegative and $index_0(A) \in \{0, 1\}$ the matrices A and A^g share some combinatorial properties for large prime numbers g. In this section, we give slight improvements of their result by expanding the set of powers g for which their result is true. We then use this set of powers to prove our main theorem in this section, Theorem 4.9, which generalizes Rothblum's result [25] on the algebraic eigenspace of a nonnegative matrix and its basic classes.

Following the notation of [6], for any real matrix A, we define a set of integers D_A (the *denominator set* of the matrix A) as follows

$$\begin{aligned} D_A &= \{d \mid \theta - \alpha = c/d, \text{ where } re^{2\pi i\theta},\ re^{2\pi i\alpha} \in \sigma(A),\ r > 0,\ c \in \mathbb{Z}^+, \\ &\quad d \in \mathbb{Z}\backslash\{0\},\ \gcd(c,d) = 1, \text{ and } |\theta - \alpha| \notin \{0,1,2,\ldots\}\}. \end{aligned}$$

The set D_A captures the denominators of those lowest terms rational numbers that represent the argument differences (normalized by a factor of $1/2\pi$) of two distinct eigenvalues of A lying on the same circle in the complex plane. In other words, if two distinct eigenvalues of A lie on the same circle in the complex plane and their argument difference is a rational multiple of 2π, then the denominator of this rational multiple in the lowest terms belongs to D_A. Note that the set D_A defined above is empty if and only if *one* of the following statements is true:

1. A has no distinct eigenvalues lying on the same circle in the complex plane.
2. The argument differences of the distinct eigenvalues of A that lie on the same circle in the complex plane are irrational multiples of 2π.

Note also that D_A is always a finite set, and 1 is never an element of D_A. Moreover, $d \in D_A$ if and only if $-d \in D_A$.

We define now the following sets of integers:

$$\begin{aligned} P_A &= \{kd \mid k \in \mathbb{Z},\ d > 0 \text{ and } d \in D_A\} && \text{(Problematic Powers of } A). \\ N_A &= \{1,2,3,\ldots\}\backslash P_A && \text{(Nice Powers of } A). \end{aligned}$$

Since D_A is finite and 1 is never an element of D_A, N_A is always an infinite set. In particular, N_A contains all the prime numbers that are larger than the maximum of D_A.

Lemma 4.1. *Let $A \in \mathbb{C}^{n\times n}$ and let λ, $\mu \in \sigma(A)$, then for all $k \in N_A$, $\lambda^k = \mu^k$ if and only if $\lambda = \mu$.*

Proof. The sufficiency is trivial. For the necessity, pick any $k \in N_A$ and suppose that $\lambda^k = \mu^k$ for some λ, $\mu \in \sigma(A)$. If $\lambda^k = \mu^k = 0$ then obviously $\lambda = \mu = 0$. Suppose that $\lambda^k = \mu^k \neq 0$. Then, in such a case, there is an $r > 0$ such that $\lambda = re^{2\pi i\theta}$, $\mu = re^{2\pi i\alpha}$ for some $\theta, \alpha \in [0,1)$. In such a case, $\lambda^k = \mu^k \Leftrightarrow r^k e^{2\pi ik\theta} = r^k e^{2\pi ik\alpha} \Leftrightarrow e^{2\pi ik(\theta-\alpha)} = 1 \Leftrightarrow k(\theta - \alpha) = m$ for some $m \in \mathbb{Z}$. Assume (with the hope of getting a contradiction) that $m \neq 0$. It is enough to consider the case when $m > 0$, since the other case is analogous. If $d = gcd(k,m)$ then we have two cases. Either $d = k$ or $d < k$. If $d = k$ then $\theta - \alpha = \frac{m}{k} \in \mathbb{Z}$. But, θ and α are in $[0,1)$. Hence $\theta - \alpha = 0 \Leftrightarrow m = 0$, a contradiction. Suppose now that $gcd(k,m) = d < k$, , then $\theta - \alpha = \frac{m/d}{k/d} \in \mathbb{Z}$ and $gcd(\frac{k}{d}, \frac{m}{d}) = 1$. Hence, $\frac{k}{d} \in D_A \Rightarrow k \in P_A \Leftrightarrow k \notin N_A$, a contradiction. □

Lemma 4.2. *Let $A \in \mathbb{C}^{n\times n}$ and let $\lambda \in \sigma(A)\backslash\{0\}$, then for all $k \in N_A$ we have $E_\lambda(A) = E_{\lambda^k}(A^k)$ and the Jordan box of λ^k in $J(A^k)$ is obtained from the Jordan box of λ in $J(A)$ by replacing λ with λ^k.*

Proof. This results follows as a special case from [19, Thereom 6.2.25]. We sketch here an alternative direct proof. Since $E_\lambda(A) \subset E_{\lambda^k}(A^k)$, it suffices to show that dim $E_\lambda(A) =$ dim $E_{\lambda^k}(A^k)$. To prove the latter statement and the claim of this lemma, it is enough to show that there is a one-to-one correspondence between the collection of Jordan blocks of λ in A and the collection of Jordan blocks of λ^k in A^k that respects the multiplicity and the order of the Jordan block. This can be done with the help of Lemma 4.1. □

The following three corollaries follow directly from Lemma 4.2 with the same proofs as in [6].

COROLLARY 4.3. *Suppose that $A \in \mathbb{R}^{n\times n}$, $index_0(A) \in \{0,1\}$, and $A^s \geq 0$ for all $s \geq m$. Then, for all $g \in N_A \cap \{m, m+1, m+2, \ldots\}$, if for some ordered partition $\kappa = (\alpha_1, \alpha_2)$ of $\{1,2,\ldots,n\}$ we have $(A^g)[\alpha_1,\alpha_2] = 0$ and $(A^g)[\alpha_2]$ is irreducible or a 1×1 zero block, then $A[\alpha_1,\alpha_2] = 0$.*

COROLLARY 4.4. *Suppose that $A \in \mathbb{R}^{n\times n}$, $index_0(A) \in \{0,1\}$, $A^s \geq 0$ for all $s \geq m$. Then, for all $g \in N_A \cap \{m, m+1, m+2, \ldots\}$, if $(A^g)_\kappa$ is in the Frobenius normal form for some ordered partition κ, then A_κ is also in the Frobenius normal form.*

COROLLARY 4.5. *Suppose that $A \in \mathbb{R}^{n\times n}$, $index_0(A) \in \{0,1\}$, and $A^s \geq 0$ for all $s \geq m$. Then, for all $g \in N_A \cap \{m, m+1, m+2, \ldots\}$, the transitive closures of the reduced graphs of A and A^g are the same.*

Recall that $v \in \mathbb{C}^n$ is a generalized eigenvector of $A \in \mathbb{C}^{n\times n}$ having order $m \geq 1$ and corresponding to $\lambda \in \mathbb{C}$ if

$$(A-\lambda I)^m v = 0 \qquad \text{but} \quad (A-\lambda I)^{m-1} v \neq 0.$$

In the following lemma, we collect some known properties of generalized eigenvectors, and then we prove a result needed for our main theorem.

LEMMA 4.6. *Let $A \in \mathbb{C}^{n\times n}$.*

(i) A vector $v \in G_\lambda(A)$ is a generalized eigenvector of order $m \geq 2$ if and only if there is a generalized eigenvector $w \in G_\lambda(A)$ of order $m-1$ such that $Av = \lambda v + w$.

(ii) Let $A \in \mathbb{C}^{n\times n}$. If v is a generalized eigenvector in $G_\lambda(A)$ of order m, then Av is also a generalized eigenvector in $G_\lambda(A)$ of order m.

(iii) Let $A \in \mathbb{C}^{n\times n}$. If v and w are generalized eigenvectors in $G_\lambda(A)$ having orders m and l, respectively, and $1 \leq l \leq m$, then $v + w$ is a generalized eigenvector that has an order m and corresponds to λ.

LEMMA 4.7. *Let $A \in \mathbb{C}^{n\times n}$ and let $\lambda \in \sigma(A)$, $\lambda \neq 0$, then $G_\lambda(A) = G_{\lambda^k}(A^k)$ for all $k \in N_A$.*

Proof. We know from Lemma 4.2 that the Jordan box corresponding to λ^k in $J(A^k)$ is obtained from the Jordan box corresponding to λ in $J(A)$ by replacing λ with λ^k. And thus, $\dim G_\lambda(A) = mult_\lambda(A) = mult_{\lambda^k}(A^k) = \dim G_{\lambda^k}(A^k)$. Hence, to prove that $G_\lambda(A) = G_{\lambda^k}(A^k)$, it is enough to show that $G_\lambda(A) \subseteq G_{\lambda^k}(A^k)$. To do that, it is enough to show that v is a generalized eigenvector of order m in $G_{\lambda^k}(A^k)$ whenever v is a generalized eigenvector of order m in $G_\lambda(A)$ for all $m \in \{1, 2, \ldots, index_\lambda(A)\}$. We prove the latter statement by induction on m, the order of v. If $m = 1$, then $v \in G_\lambda(A)$ is an ordinary eigenvector of A corresponding to λ. Hence, $Av = \lambda v$ which implies $A^k v = \lambda^k v$. And thus, $v \in G_{\lambda^k}(A^k)$ is a generalized eigenvector of A^k of order 1. Suppose that for all $1 \leq l < m$ whenever $v \in G_\lambda(A)$ is a generalized eigenvector of order l, then $v \in G_{\lambda^k}(A^k)$ is a generalized eigenvector of A^k of order l. Let $v \in G_\lambda(A)$ be a generalized eigenvector of order m. By Lemma 4.6 (i), there is a generalized eigenvector $w \in G_\lambda(A)$ of order $m-1$ such that $Av = \lambda v + w$. And thus,

$$\begin{aligned} Av &= \lambda v + w,\ A^2 v = \lambda^2 v + \lambda w + Aw,\ A^3 v = \lambda^3 v + \lambda^2 w + \lambda A w + A^2 w, \ldots, \\ A^k v &= \lambda^k v + \lambda^{k-1} w + \lambda^{k-2} A w + \cdots + \lambda A^{k-2} w + A^{k-1} w. \end{aligned}$$

By Lemma 4.6 (ii), the vectors $Aw, A^2w, \cdots, A^{k-1}w$ are all generalized eigenvectors in $G_\lambda(A)$ having order $m-1$. Hence, by Lemma 4.6 (iii), the vector $\lambda^{k-1}w + \lambda^{k-2}Aw + \cdots + \lambda A^{k-2}w + A^{k-1}w$ is a generalized eigenvector in $G_\lambda(A)$ having order $m-1$. By the induction hypothesis, the vector $\lambda^{k-1}w + \lambda^{k-2}Aw + \cdots + \lambda A^{k-2}w + A^{k-1}w$ is a generalized eigenvector in $G_{\lambda^k}(A^k)$ of order $m-1$. But, in such a case Lemma 4.6 (i), implies that v is a generalized eigenvector in $G_{\lambda^k}(A^k)$ of order m. □

Rothblum [25, Theorem 3.1] proved the following result.

THEOREM 4.8. *Let $A \in \mathbb{R}^{n\times n}$ be nonnegative and let $\mathcal{N}(A-\rho(A)I)^k$ with $k = index_{\rho(A)}(A)$ be the algebraic eigenspace corresponding to $\rho(A)$. Assume that A has m basic classes $\alpha_1, \ldots, \alpha_m$. Then, $k = m$, and the algebraic eigenspace $\mathcal{N}(A-\rho(A)I)^m$ contains nonnegative vectors $v^{(1)}, \cdots, v^{(m)}$, such that $v_j^{(i)} > 0$ if and only if the index j has access to α_i in $G(A)$, the graph of A. Furthermore, any such collection is a basis of $\mathcal{N}(A-\rho(A)I)^m$.*

We now show that this theorem holds as well for eventually nonnegative matrices A whose index is zero or one.

THEOREM 4.9. *Suppose that $A \in \mathbb{R}^{n\times n}$ is eventually nonnegative with $index_0(A) \in \{0,1\}$ and let $\mathcal{N}(A-\rho(A)I)^k$ with $k = index_{\rho(A)}(A)$ be the algebraic eigenspace corresponding to $\rho(A)$. Assume that A has m basic classes $\alpha_1, \ldots, \alpha_m$. Then $k = m$, and the algebraic eigenspace $\mathcal{N}(A-\rho(A)I)^m$ contains nonnegative vectors $v^{(1)}, \cdots, v^{(m)}$, such that $v_j^{(i)} > 0$ if and only if the index j has access to α_i in $G(A)$, the graph of A. Furthermore, any such collection is a basis of $\mathcal{N}(A-\rho(A)I)^m$.*

Proof. Since A is eventually nonnegative, it follows that there is $p \in N_A$ such that $A^s \geq 0$ for all $s \geq p$. Let $k' = index_{\rho(A^p)}(A^p)$ and let $\kappa = (\alpha_1, \cdots, \alpha_{m'})$ be an ordered partition of $\{1, 2, \ldots, n\}$ that gives the Frobenius normal form of A^p. By Theorem 4.8, $k' = m'$ and the algebraic eigenspace $\mathcal{N}(A^p - \rho(A^p)I)^{k'}$ contains nonnegative vectors $v^{(1)}, \cdots, v^{(m')}$, such that $v_j^{(i)} > 0$ if and only if the index j has access to α_i in $G(A^p)$. By Lemma 4.2, $k' = k$. Moreover, Corollary 4.4 implies that $m' = m$ and the ordered partition κ also gives the Frobenius normal form of A. Hence, $k = m$ and the classes of A are the same as the classes of A^p. Moreover, we know from Lemma 4.7 that $\mathcal{N}(A-\rho(A)I)^k = \mathcal{N}(A^p-\rho(A^p)I)^k$. Thus, $v^{(1)}, \cdots, v^{(m)}$ is a basis of $\mathcal{N}(A-\rho(A)I)^k$. Furthermore, we claim that j has access to α_i in $G(A^p)$ if and only if j has access to α_i in $G(A)$. To prove the latter claim, let β denote the class to which the index j belongs and consider the reduced graphs of A and A^p. By Corollary 4.5, the transitive closures of the reduced graphs of A and A^p are the same. Hence, the reduced graphs of A and A^p have the same access relations. Thus, β has access to α_i in the reduced graph of A if and only if β has access to α_i in the reduced graph of A^p. Since j communicates with any vertex in β, it follows that j has access to α_i in $G(A^p)$ if and only if j has access to α_i in $G(A)$, and thus, the theorem holds. □

COROLLARY 4.10. *Suppose that $A \in \mathbb{R}^{n\times n}$ is an eventually nonnegative matrix with $index_0(A) \in \{0,1\}$. Then, there is a positive eigenvector corresponding to $\rho(A)$ if and only if the final classes of A are exactly its basic ones.*

COROLLARY 4.11. *Suppose that $A \in \mathbb{R}^{n\times n}$ is an eventually nonnegative matrix with $index_0(A) \in \{0,1\}$. Then, there are positive right and left eigenvectors corresponding to $\rho(A)$ if and only if all the classes of A are basic and final, i.e., A is permutationally similar to a direct sum of irreducible matrices having the same spectral radius.*

5. Relations between sets. We present in this section inclusion relations between the different sets defined in the introduction.

We begin by mentioning that

$$\text{PFn} = \{\text{Eventually Positive Matrices}\}. \tag{5.1}$$

This equality follows from [20, Theorem 1], [23, Theorem 2.2], and [36, Theorem 4.1 and Remark 4.2].

The proof of the following lemma can be found in [23]. Here, we have added the necessary hypothesis of having at least one nonzero eigenvalue or equivalently having a positive spectral radius.

LEMMA 5.1. *If $A \in \mathbb{R}^{n\times n}$ is eventually nonnegative and has at least one nonzero eigenvalue, then, both matrices A and A^T possess the Perron-Frobenius property, i.e., $A \in$ WPFn.*

We illustrate with the following example the need of at least one nonzero eigenvalue in the hypothesis of Lemma 5.1.

EXAMPLE 5.2. It is essential for an eventually nonnegative matrix A to have a nonzero eigenvalue for A to be in WPFn. Let $A = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$, then $A^2 = 0$. Hence, A is eventually nonnegative. But, 0 is the only eigenvalue of A, the Jordan canonical form of A is $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, and all the ordinary eigenvectors of A are of the form $[\alpha, -\alpha]^T$ for some $\alpha \neq 0$, $\alpha \in \mathbb{R}$. Therefore, A does not possess the strong Perron-Frobenius property nor the Perron-Frobenius property. To avoid such a situation, we have to stipulate that at least one of the eigenvalues of A is nonzero or equivalently $\rho(A) > 0$.

COROLLARY 5.3. *Not all eventually nonnegative matrices are in WPFn.*

In fact, we can see from Example 5.2, Lemma 5.1, and Corollary 5.3, that all eventually nonnegative matrices are inside WPFn with the exception of nilpotent matrices. In fact, the set of nonnilpotent eventually nonnegative matrices is a proper subset of WPFn as we show in the following proposition.

PROPOSITION 5.4. *The collection of eventually nonnegative matrices with at least one nonzero eigenvalue is properly contained in WPFn.*

Proof. It suffices to find a matrix A in WPFn which is not eventually nonnegative. Consider the matrix $A = J \oplus [-1]$ where J is the matrix of dimension $(n-1)$ having all its entries equal to 1. Then, $A^k = [(n-1)^{(k-1)} J] \oplus [(-1)^k]$. Clearly, A is not eventually nonnegative because the (n, n)-entry of A keeps on alternating signs. However, $A \in$ WPFn since $\rho(A) = n - 1$ and there is a semipositive vector $v = (1 \ \cdots \ 1 \ 0)^T \in \mathbb{R}^n$ satisfying $v^T A = \rho(A) v^T$ and $Av = \rho(A)v$. ☐

Thus, Proposition 5.4 tells us that if we exclude nilpotent matrices from the collection of eventually nonnegative matrices, then still we do not cover all of WPFn. Hence, Proposition 5.4 establishes that all the containments are proper in the following statement:

$$\begin{aligned} \text{PFn} &= \{\text{Eventually Positive Matrices}\} \\ &\subset \{\text{Nonnilpotent eventually nonnegative matrices}\} \\ &\subset \text{WPFn}. \end{aligned}$$

Moreover, it turns out that an irreducible matrix in WPFn does not have to be eventually nonnegative as the following example inspired by [6, Example 3.1] shows.

EXAMPLE 5.5. Let

$$A = \begin{bmatrix} 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 \end{bmatrix}, \ B = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 \end{bmatrix}, \text{ and } C = \begin{bmatrix} 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Note that A is an irreducible matrix. Also, note that $\rho(A) = 2$ and that if $v = [2, 2, 1, 1]^T$ and $w = [1, 1, 0, 0]^T$, then $Av = \rho(A)v$ and $w^T A = \rho(A) w^T$. Thus, A

is an irreducible matrix in WPFn. Furthermore, it is easy to see that $A = B + C$ and that $BC = CB = C^2 = 0$. Hence, $A^j = B^j$ for all $j \geq 2$. But B is not eventually nonnegative since the lower right 2×2 diagonal block of B keeps on alternating signs. Thus, A is not eventually nonnegative.

6. Matrices that are eventually in WPFn and PFn. As we have seen, nonnilpotent eventually nonnegative matrices have the Perron-Frobenius property. It is natural then to ask, what can we say of matrices whose powers eventually belong to WPFn (or PFn). We show in this short section that these matrices must belong to WPFn (or PFn).

THEOREM 6.1. *$A \in$ WPFn if and only if for some integer k_0, $A^k \in$ WPFn, for all $k \geq k_0$.*

Proof. Suppose that $A \in$ WPFn. For any $\lambda \in \sigma(A)$, $\lambda \neq 0$ and all $k \geq 1$, we have $E_\lambda(A) \subseteq E_{\lambda^k}(A^k)$. In particular, this is true for $\lambda = \rho(A) > 0$. Using the fact that $\rho(A^k) = (\rho(A))^k$, we see that $E_{\rho(A)}(A) \subseteq E_{\rho(A)^k}(A^k)$. Thus, if A has the Perron-Frobenius property, then so does A^k for all $k \geq 1$. Likewise, $(A^T)^k$ has the Perron-Frobenius property for all $k \geq 1$. Thus, $A^k \in$ WPFn for all $k \geq 1$. Conversely, suppose that there is a positive integer k_0 such that $A^k \in$ WPFn for $k \geq k_0$. Since the eigenvalues of A are the k^{th} roots of the eigenvalues of A^k for all $k \geq k_0$, it follows that $0 \neq \rho(A) \in \sigma(A)$. Moreover, by picking $k \in N_A \cap \{k_0, k_0+1, k_0+2, \ldots\}$ ("nice powers" k of A that are larger than k_0), we have $E_{\rho(A)}(A) = E_{\rho(A)^k}(A^k)$ (this follows from Lemma 4.2). Hence, we can choose a nonnegative eigenvector of A corresponding $\rho(A)$. So, A has the Perron-Frobenius property. Similarly, A^T has the Perron-Frobenius property. Thus, $A \in$ WPFn. □

Similarly, we obtain the following result.

THEOREM 6.2. *$A \in$ PFn if and only if for some integer k_0, $A^k \in$ PFn, for all $k \geq k_0$.*

7. Similarity matrices preserving the Perron-Frobenius property. If S is a positive diagonal matrix or a permutation matrix then clearly $S^{-1}AS$ possesses the Perron-Frobenius property whenever A does. This observation leads to the following question: which similarity matrices S preserve the Perron-Frobenius property, the strong Perron-Frobenius property, or being in WPFn, or in PFn? In other words, for which similarity transformations is WPFn or PFn invariant? We first prove a preliminary lemma that leads to answering these questions.

LEMMA 7.1. *Let S be an $n \times n$ real matrix which has a positive entry and a negative entry. If S is of rank one but not expressible as xy^T where x is a nonnegative vector or is of rank two or more, then there is a positive vector $v \in \mathbb{R}^n$ such that Sv has a positive entry and a negative entry.*

Proof. If S is a rank-one matrix with the given property, then S is expressible as xy^T , where x is a vector which has a positive entry and a negative entry. Choose any positive vector v such that $y^Tv \neq 0$. Then Sv, being a nonzero multiple of x, clearly has a positive entry and a negative entry.

Suppose that S is of rank two or more. If S has a column which has a positive entry and a negative entry, say, the kth column, then take v to be the positive vector in $\mathbb{R}^n$ whose kth entry is 1 and all of whose other entries equal ϵ. It is readily seen that for $\epsilon > 0$ sufficiently small, Sv has a positive entry and a negative entry. It remains to consider the case when every nonzero column of S is either semipositive or seminegative. Because S is of rank two or more, it is possible to choose two linearly independent columns of S, with one semipositive and the other seminegative; say the

jth column is semipositive and the kth column is seminegative. If the jth column has a zero entry such that the corresponding entry for the kth column is negative, then clearly $S(e_j + \delta e_k)$ (where e_i denotes the ith standard unit vector of $\mathbb{R}^n$) has a positive entry and a negative entry for sufficiently small $\delta > 0$, hence so does the vector Sv where v is the positive vector of $\mathbb{R}^n$ with 1 at its jth entry, δ at its kth entry and ϵ at its other entries, where $\epsilon > 0$ is sufficiently small. Similarly, if the kth column has a zero entry such that the corresponding entry for the jth column is positive, then by a similar argument we are also done. So, the jth and the kth columns of S have 0s at exactly the same positions. Consider $S((1-\lambda)e_j + \lambda e_k)$. Let λ_0 be the largest $\lambda \in [0, 1]$ such that $S((1-\lambda)e_j + \lambda e_k)$ is nonnegative. Because we assume that the jth and the kth columns of S are linearly independent, it is clear that $S((1-\lambda_0)e_j + \lambda_0 e_k)$ is in fact semipositive. Choose $\lambda_1 > \lambda_0$, sufficiently close to λ_0. Then $S((1-\lambda_1)e_j + \lambda_1 e_k)$ has a positive entry and a negative entry. Now let v be the positive vector in $\mathbb{R}^n$ whose jth entry is $1-\lambda_1$, whose kth entry is λ_1, and all of whose other entries are ϵ. Then for $\epsilon > 0$ sufficiently small Sv has a positive entry and a negative entry. □

We call a matrix S *monotone* if $S \in GL(n, \mathbb{R})$ and S^{-1} is nonnegative.

THEOREM 7.2. *For any $S \in GL(n, \mathbb{R})$, the following statements are equivalent:*

(*i*) *Either S or $-S$ is monotone.*

(*ii*) *$S^{-1}AS$ has the strong Perron-Frobenius property for all matrices A having the strong Perron-Frobenius property.*

Proof. Suppose (*i*) is true. Assume without loss of generality that S is monotone. If A is a matrix with the strong Perron-Frobenius property and v is a right Perron-Frobenius eigenvector of A, then $S^{-1}v$ is an eigenvector of $S^{-1}AS$ corresponding to $\rho(A)$. The nonsingularity of S implies that none of the rows of S^{-1} is 0. Therefore, $S^{-1}v$ is a positive vector. Also, $\rho(A)$ is a simple positive and strictly dominant eigenvalue of $S^{-1}AS$ since $S^{-1}AS$ and A have the same characteristic polynomial. This shows that (*i*) $\Rightarrow$ (*ii*). Conversely, suppose (*i*) is not true, i.e., S and $-S$ are both not monotone. Then, in such a case, S^{-1} must have a positive entry and a negative entry. By Lemma 7.1, there is a positive vector v such that $S^{-1}v$ has a positive entry and a negative entry. For any scalar $\rho > 0$, we can construct the matrix $A = (\rho/v^T v)vv^T \in$ PFn, having v as a right Perron-Frobenius eigenvector. Moreover, for such a matrix A, we have $E_{\rho(A)}(A) = Span\{v\}$. Since the eigenvectors in $E_{\rho(A)}(S^{-1}AS)$ are of the form $S^{-1}w$ for some eigenvector $w \in E_{\rho(A)}(A)$, it follows that $E_{\rho(A)}(S^{-1}AS)$ does not have a positive vector. Thus, $S^{-1}AS$ does not have the strong Perron-Frobenius property. Hence, (*ii*) is not true, which shows that (*ii*) $\Rightarrow$ (*i*). □

The following results follow in the same manner.

THEOREM 7.3. *For any $S \in GL(n, \mathbb{R})$, the following statements are equivalent:*

(*i*) *Either S or $-S$ is monotone.*

(*ii*) *$S^{-1}AS$ has the Perron-Frobenius property for all matrices A having the Perron-Frobenius property.*

COROLLARY 7.4. *For any $S \in GL(n, \mathbb{R})$, the following statements are equivalent:*

(*i*) *S and S^{-1} are either both nonnegative or both nonpositive.*

(*ii*) *$S^{-1}AS \in PFn$ for all $A \in PFn$.*

(*iii*) *$S^{-1}AS \in WPFn$ for all $A \in WPFn$.*

8. Spectral decomposition and the Perron-Frobenius property. In this section, we give a characterization of all matrices in PFn and some matrices in WPFn in terms of the positivity or nonnegativity of their spectral projectors.

We begin with the following known result. We omit its proof since it can be derived from the usual spectral decomposition that can be found, e.g., in [7, page 27] or [31, pages 114, 225] and its method of proof is similar to that of [36, Theorem 3.6].

THEOREM 8.1. *If $A \in \mathbb{C}^{n\times n}$ has d distinct eigenvalues $|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_d|$ then A has a decomposition: $A = \lambda_1 P + Q$ satisfying the following:*

(i) P is the projection matrix onto $G_{\lambda_1}(A)$ along $\bigoplus_{j=2}^{d} G_{\lambda_j}(A)$.
(ii) $PQ = QP$.
(iii) $\rho(Q) \leq \rho(A)$.
(iv) If $index_{\lambda_1}(A) = 1$ then $PQ = QP = O$.

THEOREM 8.2. *The following statements are equivalent:*

(i) $A \in PFn$.
(ii) $\rho(A)$ is an eigenvalue of A and in the spectral decomposition $A = \rho(A)P + Q$ we have $P > 0$, rank $P = 1$ and $\rho(Q) < \rho(A)$.

Proof. Suppose that $A \in$ PFn. Then, both A and A^T have a positive (or negative) eigenvector corresponding to a simple, positive, and strictly dominant eigenvalue $\rho = \rho(A)$. Let us call them v and w, respectively, and they are normalized so that $v^T w = 1$. We then have that in the spectral decomposition $A = \rho(A)P + Q$, we have $\rho(Q) < \rho(A)$. Moreover, since $G_\rho(A)$ is one-dimensional, the rank of the projection matrix P that projects onto $G_\rho(A)$ must be equal to 1. In fact, we have $P = vw^T$. Since $w^T v = 1$, it follows that the vectors v and w are either both positive or both negative. Therefore, $P = vw^T > O$. Conversely, suppose that $\rho = \rho(A)$ is an eigenvalue of A and that in the spectral decomposition $A = \rho(A)P + Q$, we have $P > 0$, rank $P = 1$ and $\rho(Q) < \rho(A)$. Since rank P =1, it follows that the algebraic multiplicity of ρ is 1. Thus, $index_\rho(A) = 1$ and by Theorem 8.1 we conclude that $PQ = QP = O$. Therefore,

$$\left(\frac{1}{\rho}A\right)^k = \left(P + \frac{1}{\rho}Q\right)^k = P^k + \left(\frac{1}{\rho}Q\right)^k = P + \left(\frac{1}{\rho}Q\right)^k,$$

and consequently,

$$\lim_{k\to\infty}\left(\frac{1}{\rho}A\right)^k = P + \lim_{k\to\infty}\left(\frac{1}{\rho}Q\right)^k = P > 0.$$

Since $\rho > 0$ and the matrix $\frac{1}{\rho}A$ is real and eventually positive, it follows that the matrix A is also real and eventually positive. By (5.1), $A \in$ PFn. ☐

The proofs of the following two results are very similar to that of Theorem 8.2, and are therefore omitted.

THEOREM 8.3. *The following statements are equivalent:*

(i) $A \in WPFn$ has a simple, positive, and strictly dominant eigenvalue.
(ii) $\rho(A)$ is an eigenvalue of A and in the spectral decomposition $A = \rho(A)P + Q$ we have $P \geq 0$, rank $P = 1$ and $\rho(Q) < \rho(A)$.

THEOREM 8.4. *Let one of the two real matrices A and A^T possess the strong Perron-Frobenius property but not the other. Then, the projection matrix P in the spectral decomposition of A satisfies the relation $P = vw^T$ where one of the vectors v and w is positive and the other has both positive and negative entries.*

COROLLARY 8.5. *If one of the two real matrices A and A^T has a Perron-Frobenius eigenpair of a strictly dominant simple positive eigenvalue and a nonnegative eigenvector but the other matrix does not, then the projection matrix P in the spectral decomposition of A has some negative entries, and rank $P = 1$.*

9. The Perron-Frobenius property and real symmetric matrices. In this section, S_n denotes the collection of $n \times n$ real symmetric matrices and e denotes the vector that consists entirely of ones. We study the boundary of the cone in S_n of maximal angle centered at ee^T (the matrix of ones) in which the nonnegativity of both the dominant eigenvalue and its corresponding eigenvector is retained. Tarazaga, Raydan, and Hurman studied this cone for $n \geq 3$ [33, Theorem 4.1] and showed that the angle of such a cone is

$$\alpha = \arccos\left(\frac{\sqrt{(n-1)^2+1}}{n}\right). \tag{9.1}$$

The authors of [33] do not claim that PFn or WPFn is a cone. In fact, it is shown by Johnson and Tarazaga [20] that PFn is not even convex. Thus, neither PFn nor WPFn is necessarily a cone. We explore this further and show that there is a curve of matrices with the strong Perron-Frobenius property extending outside the cone centered at ee^T and making an angle α given in (9.1).

Proposition 9.1. *The maximal subset of S_n ($n \geq 3$) for which there is a nonnegative Perron-Frobenius eigenpair extends outside the cone centered at ee^T whose angle α is given by (9.1).*

Proof. Consider a matrix A in S_n ($n \geq 3$) of the form:

$$A = A(x) = \begin{bmatrix} x & x & x & \cdots & x \\ x & 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ x & 1 & \cdots & 1 & 1 \\ x & 1 & \cdots & 1 & x \end{bmatrix}$$

where x is a positive scalar. Obviously, the matrix A is a positive matrix in S_n, and thus, it possesses the strong Perron-Frobenius property for every positive scalar x.

We will show that there exists a $\delta > 0$ such that $Angle(A, ee^T) > \alpha$ (i.e., $\cos(A, ee^T) < \cos\alpha$) whenever $0 < x < \delta$. First, let us compute the cosine of the angle between A and ee^T. Let a_{ij} denote the (i,j)-entry of A. Then,

$$\cos(A, ee^T) = \frac{\sum_{i,j} a_{ij}}{n\sqrt{\sum_{i,j}(a_{ij})^2}} = \frac{2nx+n^2-2n}{n\sqrt{2nx^2+n^2-2n}}$$

And thus, $\cos(A, ee^T) < \cos\alpha$ if and only if $\frac{2nx+n^2-2n}{n\sqrt{2nx^2+n^2-2n}} < \frac{\sqrt{(n-1)^2+1}}{n}$.
Define the functions $f_n(x)$ ($n \geq 3$) by:

$$f_n(x) = \frac{\sqrt{(n-1)^2+1}}{n} - \frac{2nx+n^2-2n}{n\sqrt{2nx^2+n^2-2n}}.$$

Then, we want to see when $f_n(x) > 0$. Note that for all $n \geq 3$:

$$f_n(0) = \frac{\sqrt{(n-1)^2+1}}{n} - \frac{n^2-2n}{n\sqrt{n^2-2n}} = \frac{\sqrt{n^2-2n+2} - \sqrt{n^2-2n}}{n} \quad > \ 0$$

By continuity of the function $f_n(x)$, there exists $\delta_n > 0$ such that $f_n(x) > 0$ whenever $0 < x < \delta_n$. Hence, when $0 < x < \delta_n$ the matrix A possesses the strong Perron-Frobenius property yet it lies outside the cone centered at ee^T with angle α. Indeed, we can define a curve $A_\gamma : [0,1) \to S_n$ of the matrices $A_\gamma(t) = A(1-t)$, which lie outside the cone centered at ee^T with angle α, for $1-\delta_n < t < 1$, while they satisfy the strong Perron-Frobenius property. Furthermore, since the eigenvalues and the eigenvectors depend continuously on the matrix entries (see, e.g., [3], [18]), it follows that there is a neighborhood of the curve $A_\gamma(t)$ defined for $1-\delta_n < t < 1$, in which the strong Perron-Frobenius property holds as well. The intersection of this neighborhood with S_n further extends the known collection of such matrices lying outside the cone mentioned above. □

10. Topological properties. In this section, we prove some topological properties of the collections of matrices with the Perron-Frobenius property and other subcollections.

The following lemma was asserted and used by Johnson and Tarazaga in the proof of [20, Theorem 2]. Its proof follows directly from [4, Theorem 27]; see also [23] for a recent proof.

LEMMA 10.1. *Let A be a matrix in $\mathbb{R}^{n\times n}$ with the Perron-Frobenius property, and let v be its right Perron-Frobenius eigenvector. If $w \in \mathbb{R}^n$, $w \neq 0$, is such that $v^T w > 0$ then for all scalars $\epsilon > 0$ the following holds:*

(i) The matrix $B = A + \epsilon v w^T$ has the Perron-Frobenius property.
(ii) $\rho(A) < \rho(B)$.
(iii) If A has the strong Perron-Frobenius property then so does B.

THEOREM 10.2. *The collection of matrices in $\mathbb{R}^{n\times n}$ with the Perron-Frobenius property is path-connected.*

Proof. Let $A \in \mathbb{R}^{n\times n}$ be any matrix with the Perron-Frobenius property. Since the collection of positive matrices is convex, it is enough to show that there is a path connecting matrix A to some positive matrix B. The proof goes as follows: connect matrix A to a matrix $\tilde{A}$ having a positive right Perron-Frobenius eigenvector and then connect $\tilde{A}$ to a positive matrix B.

If A has a positive right Perron-Frobenius eigenvector then define $\tilde{A} = A$, otherwise, consider $J(A)$, the Jordan canonical form of A. We know that $A = VJ(A)V^{-1}$ where $V = [v\ w_2\ w_3 \cdots w_n]$ and v is a right Perron-Frobenius eigenvector of A. For every scalar $t \geq 0$, we construct the vector v_t by replacing the zero entries of v by t, and we construct a new matrix $V_t = [v_t\ w_2\ w_3 \cdots w_n]$. Since $V_0 = V \in GL(n,\mathbb{C})$ and since $GL(n,\mathbb{C})$ is an open subset of $\mathbb{C}^{n\times n}$, there is a positive scalar δ such that whenever $0 \leq t \leq \delta$ we have $V_t \in GL(n,\mathbb{C})$. Define $A_t = V_t J(A) V_t^{-1}$ for $0 \leq t \leq \delta$. Then, A_t is a path of complex matrices having a positive dominant eigenvalue $\rho(A)$ with a corresponding nonnegative eigenvector v_t. Moreover, v_t is positive for all $0 < t \leq \delta$. Note that $A_t = C_t + iD_t$ where C_t and D_t are paths of real matrices. Since $A_t v_t = C_t v_t + iD_t v_t = \rho(A) v_t \in \mathbb{R}^n$ for all $0 \leq t \leq \delta$, it follows that $D_t v_t = 0$ and that $C_t v_t = \rho(A) v_t$ for all $0 \leq t \leq \delta$. Moreover, $C_0 = A_0 = A$. Hence, C_t is a path of real matrices connecting A to C_δ, and each matrix C_t has a positive dominant eigenvalue $\rho(A)$ with a corresponding nonnegative eigenvector v_t, therefore having the Perron-Frobenius property. Let $\tilde{A} = C_\delta$ and let v_δ be its corresponding positive eigenvector.

Let w be any positive vector. For all scalars $\epsilon \geq 0$, define the path of real matrices $K_\epsilon = \tilde{A} + \epsilon v_\delta w^T$. By Lemma 10.1, K_ϵ possesses the Perron-Frobenius property for all $\epsilon \geq 0$. Since $v_\delta w^T$ is a positive matrix, K_ϵ is positive for large values of ϵ. Hence, there is a positive real number M such that K_M is positive. Let $B = K_M$. Hence, K_ϵ is a path connecting $\tilde{A}$ to B. □

Similarly, we have the following result.

THEOREM 10.3. *The collection of matrices in $\mathbb{R}^{n\times n}$ with the strong Perron-Frobenius property is path-connected.*

COROLLARY 10.4. *PFn is simply connected.*

Proof. Johnson and Tarazaga proved in [20, Theorem 2] that PFn is path-connected. Thus, it is enough to show that any loop in PFn can be shrunk to a point. Let $A_t : [0,1] \rightarrow$ PFn be a loop of matrices in PFn. For all $0 \leq t \leq 1$, let v_t and w_t be respectively the right and the left Perron-Frobenius unit eigenvectors of A_t. Also, for all scalars $\epsilon \geq 0$, define the loop $B_t^\epsilon = A_t + \epsilon v_t w_t^T$. By Lemma 10.1, the loop B_t^ϵ is in PFn for all scalars $\epsilon \geq 0$. Note that for large values of ϵ the loop B_t^ϵ is a loop of positive matrices. Hence, A_t can be continuously deformed to a loop that can be shrunk to a point. □

COROLLARY 10.5. *The collection of matrices in $\mathbb{R}^{n\times n}$ with the strong Perron-Frobenius property is simply connected.*

PROPOSITION 10.6. *The closure* $\overline{\text{WPFn}} = \text{WPFn} \cup \{$*nilpotent matrices with a pair of right and left nonnegative eigenvectors*$\}$.

Proof. Since the eigenvalues and eigenvector entries are continuous functions of the matrix entries, it follows that for any matrix A in $\overline{\text{WPFn}}$ we have $\rho(A) \geq 0$ and A has a pair of left and right nonnegative eigenvectors corresponding to $\rho(A)$. If $\rho(A) = 0$ then A is nilpotent with a pair of right and left nonnegative eigenvectors, otherwise A is in WPFn. Conversely, suppose that A is in WPFn or A is a nilpotent matrix with a pair of right and left nonnegative eigenvectors. If A is in WPFn then obviously A is in $\overline{\text{WPFn}}$. If A is a nilpotent matrix with a pair of right and left nonnegative eigenvectors v and w, respectively, then A has a Jordan canonical form $A = VJ(A)V^{-1}$, where V and V^{-1} are real matrices, all the Jordan blocks in $J(A)$ are of the form $J_s(0)$ for some $s \in \{1, \ldots, n\}$, the i^{th} column of V is v, and the j^{th} row of V^{-1} is w^T for some $i, j \in \{1, \ldots, n\}$. Let e_i denote the ith canonical vector. For every positive scalar ϵ, let $J_\epsilon = J(A) + \epsilon(e_i e_i^T + e_j e_j^T)$, and $A_\epsilon = VJ_\epsilon V^{-1}$. Note that $A_\epsilon v = \epsilon v$ and $w^T A_\epsilon = \epsilon w^T$. Hence, $A_\epsilon \in$ WPFn for all $\epsilon > 0$. Moreover, A_ϵ converges to A as $\epsilon \rightarrow 0$. □

LEMMA 10.7. *For any semipositive vector v_1 and for any scalar $\epsilon > 0$, there is an orthogonal matrix Q such that $||Q - I||_2 < \epsilon$ and $Qv_1 > 0$.*

Proof. Assume without loss of generality that v_1 is a unit 2-norm vector. If v_1 is a positive vector then let $Q = I$, otherwise pick any scalar $\epsilon > 0$ and replace the zero entries of v_1 by positive entries that are small enough then normalize so that the obtained vector, say $\tilde{v}$, is a positive unit 2-norm vector and $||\tilde{v} - v_1||_2 < \epsilon/n$. Let $S = I - v_1 v_1^T$ be the projection matrix onto $v_1^\perp$, the hyperplane orthogonal to v_1, and let $v_2 = S\tilde{v}/||S\tilde{v}||_2$. Then, v_2 is a unit 2-norm vector which is orthogonal to v_1. Moreover, $\tilde{v}$ lies in the 2-dimensional plane determined by v_1 and v_2. Let $\theta = \text{Angle}(v_1, \tilde{v}) = \arccos(v_1^T \tilde{v})$. Extend $\{v_1, v_2\}$ to an orthonormal basis $\{v_1, v_2, \cdots, v_n\}$ of $\mathbb{R}^n$. Define Q to be the Givens rotation (see, e.g, [13]) by the angle θ in the 2-dimensional plane determined by v_1 and v_2. Then, $\tilde{v} = Qv_1$, $||Qv_2 - v_2||_2 = ||Qv_1 - v_1||_2 < \epsilon/n$, and $Qv_i = v_i$ for all $i \geq 3$. Therefore, $Qv_1 = \tilde{v} > 0$ and $||Q - I||_2 = sup_{||x||_2=1}||(Q - I)x||_2 < \epsilon$. □

PROPOSITION 10.8. *Every normal matrix in WPFn is the limit of normal matrices in PFn.*

Proof: Let A be a normal matrix in WPFn. Then, $A = VKV^T$ where V is an orthogonal matrix, $K = [\rho(A)] \oplus M_2 \cdots \oplus M_k$, and each M_i for $i = 2, \ldots, k$ is a real 1×1 block or a positive scalar multiple of a real orthogonal 2×2 block; see, e.g., [18, Theorem 2.5.8]. Moreover, one of the columns of V, say the first column which we denote by v, is both a right and a left Perron-Frobenius eigenvector of A. For any scalar $\epsilon > 0$, consider the matrix $B = V[[\rho(A) + \epsilon] \oplus \cdots \oplus M_k]V^T$ which has a simple positive and strictly dominant eigenvalue $\rho(A) + \epsilon$. Note that B converges to A as $\epsilon \to 0$. By Lemma 10.7, there is an orthogonal matrix Q such that $Qv > 0$ and $||Q - I||_2 < \epsilon$. Let $C = QBQ^T$. Then, C is a normal matrix having $\rho(A) + \epsilon$ as simple positive and strictly dominant eigenvalue. Moreover, C satisfies the following vector equalities $CQv = (\rho(A) + \epsilon)Qv$, and $(Qv)^T C = (\rho(A) + \epsilon)(Qv)^T$. Therefore, C is a normal matrix in PFn. Furthermore,

$$\begin{aligned} ||C - A||_2 &\leq ||C - B||_2 + ||B - A||_2 \\ &= ||QBQ^T - B||_2 + ||B - A||_2 \\ &= ||QB - BQ||_2 + ||B - A||_2 \\ &\leq ||QB - B||_2 + ||B - BQ||_2 + ||B - A||_2 \\ &\leq 2||B||_2 \, ||Q - I||_2 + ||B - A||_2 \to 0 \text{ as } \epsilon \to 0. \qquad \square \end{aligned}$$

We remark that Proposition 10.8 holds for more general matrices, namely for spectral matrices; see, e.g., [19, §1.6, problems 32 and 33]. In other words, we have shown that every spectral matrix in WPFn is the limit of spectral matrices in PFn. Since any normal matrix is spectral, Proposition 10.8 is just a special case.

11. Singular values and singular vectors. We explore in this section the application of our results on the properties of the singular values and singular vectors of the matrices studied in this paper.

THEOREM 11.1. *If A is a normal matrix in PFn, then the maximum singular value is strictly larger than the other singular values and its corresponding right and left singular vectors are positive, and equal to each other.*

Proof. Let $A = U\Sigma V^T = [u_1 \ u_2 \cdots u_n]$ rm $\text{diag}(\sigma_1, \sigma_2, \ldots, \sigma_n)$ $[v_1 \ v_2 \cdots v_n]^T$ be the singular value decomposition of A, where U and V orthogonal, and $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n \geq 0$ are the singular values of A. Note that v_1 and u_1 are, respectively, the right and the left singular vectors corresponding to σ_1, i.e., $Av_1 = \sigma_1 u_1$ and $A^T u_1 = \sigma_1 v_1$. Since $A \in$ PFn, then both A and A^T are commuting eventually positive matrices. Hence, $A^T A$ is eventually positive, and thus, it possesses the strong Perron-Frobenius property. But,

$$A^T A = AA^T = V\Sigma^2 V^T = [v_1 \ v_2 \cdots v_n] \ \text{diag}(\sigma_1^2, \sigma_2^2, \ldots, \sigma_n^2) \ [v_1 \ v_2 \cdots v_n]^T$$

is the Jordan decomposition of $A^T A = AA^T$. Therefore, v_1 must be positive and $\sigma_1^2 > \sigma_i^2 \geq 0$ for all $i \in \{2, \ldots, n\}$. $\square$

THEOREM 11.2. *If A is a normal matrix in WPFn, then its right and left singular vectors corresponding to its maximum singular value are equal and nonnegative.*

Proof. If A is a normal matrix in WPFn, then by Proposition 10.8 there is a sequence of normal matrices $\{A_k\}_{k=1}^{\infty} \subset$ PFn that converges to A. For each $k = 1, 2, \ldots$, the matrices A_k and A_k^T are commuting eventually positive matrices. And just like in the proof of Proposition 11.1, the matrix $A_k^T A_k$ possesses the strong Perron-Frobenius property and its (positive) Perron-Frobenius eigenvector, say v_k, is the left and right singular vector of the maximum singular value of A_k. By continuity of eigenvector entries as functions of matrix entries and since $A_k^T A_k \to A^T A$ and $A_k A_k^T \to AA^T$ as $k \to \infty$, it follows that the positive unit 2-norm vector v_k which is an eigenvector of $A_k^T A_k$ and of $A_k A_k^T$ converges to some nonnegative unit 2-norm vector v which is an eigenvector of $A^T A$ and AA^T, i.e., to the right and left singular vector of A corresponding to the maximum singular value. $\square$

Example 11.3. If a matrix in PFn or WPFn is not normal or, equivalently, not unitarily diagonalizable, then the singular vectors may have positive and negative entries. For example, consider the matrix

$$C = \frac{1}{4}\begin{bmatrix} 11 & 30 & -9 \\ 7 & 2 & 23 \\ 7 & 30 & -5 \end{bmatrix}.$$

The matrix C is a diagonalizable matrix in PF3, but it is not unitarily diagonalizable. In fact, the Jordan decomposition of C is given by $C = XJ(C)X^{-1}$, where

$$X = \begin{bmatrix} 1 & 1 & -3 \\ 1 & -1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \quad J(C) = \begin{bmatrix} 8 & 0 & 0 \\ 0 & -7 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad X^{-1} = \frac{1}{4}\begin{bmatrix} 1 & 2 & 1 \\ 0 & -2 & 2 \\ -1 & 0 & 1 \end{bmatrix}.$$

The singular value decomposition of C yields that the right singular vector corresponding to the maximum singular value is $[-0.2761, \ -0.9311, \ 0.2385]^T$ and the corresponding left singular vector is $[-0.7289, \ 0.0372, \ -0.6836]^T$, each of which has a negative entry and a positive entry. Similarly, by taking direct sums of matrix C with positive matrices, one can find counter-examples in WPFn in which the right and left singular vectors corresponding to the maximum singular value have positive and negative entries.

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