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Perron-Frobenius property converging to a given
matrix with the Perron-Frobenius property**

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PATHS OF MATRICES WITH THE STRONG PERRON-FROBENIUS PROPERTY CONVERGING TO A GIVEN MATRIX WITH THE PERRON-FROBENIUS PROPERTY*

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Abstract. A matrix is said to have the Perron-Frobenius property (strong Perron-Frobenius property) if its spectral radius is an eigenvalue (a simple positive and strictly dominant eigenvalue) with a corresponding semipositive (positive) eigenvector. It is known that a matrix A with the Perron-Frobenius property can always be the limit of a sequence of matrices $A(\varepsilon)$ with the strong Perron-Frobenius property such that $\|A - A(\varepsilon)\| \leq \varepsilon$. In this note we study the form that the parameterized matrices $A(\varepsilon)$ and their spectral characteristics can take. We show that it is possible to have $A(\varepsilon)$ cubic, its spectral radius quadratic and the corresponding positive eigenvector linear (all as functions of ε); further, if the spectral radius of A is simple, positive and strictly dominant, then $A(\varepsilon)$ can be taken to be quadratic and its spectral radius linear (in ε). Two other cases are discussed: when A is normal it is shown that the sequence of approximating matrices $A(\varepsilon)$ can be written as a quadratic polynomial in trigonometric functions, and when A has semipositive left and right Perron-Frobenius eigenvectors and $\rho(A)$ is simple, the sequence $A(\varepsilon)$ can be represented as a polynomial in trigonometric functions of degree at most six.

Key words. Perron-Frobenius property, generalization of nonnegative matrices, eventually nonnegative matrices, eventually positive matrices, perturbation.

AMS subject classifications. 15A48

1. Introduction. A real matrix A is called nonnegative (respectively, positive) if it is entry-wise nonnegative (respectively, positive) and we write $A \geq 0$ (respectively, $A > 0$). This notation and nomenclature is also used for vectors. A column or a row vector v is called *semipositive* if v is nonzero and nonnegative. Likewise, if v is nonzero and entry-wise nonpositive then v is called *seminegative*. We denote the spectral radius of a matrix A by $\rho(A)$. We say that a real square matrix A has the *Perron-Frobenius (P-F) property* if $Av = \rho(A)v$ for some semipositive vector v , called a right P-F eigenvector, or simply a P-F eigenvector. We call a semipositive w a left P-F eigenvector if $A^T w = \rho(A)w$. Moreover, we say that A possesses the *strong P-F property* if A has a simple, positive, and strictly dominant eigenvalue with a positive eigenvector. Further, define the sets WPFn (respectively, PFn) of $n \times n$ real matrices A such that A and A^T have the P-F property (respectively, the strong P-F property); see, e.g., [2], [3], [11], [14], where these concepts are studied and used. The P-F property is historically associated with nonnegative matrices; see the seminal papers by Perron [12] and Frobenius [5] or the classic books [1], [7], [15], for many applications.

In [2, Theorem 6.15], it is shown that for any matrix A with the P-F property, and any $\varepsilon > 0$, there exists a matrix $A(\varepsilon)$ with the strong P-F property such that $\|A - A(\varepsilon)\| \leq \varepsilon$. In the same paper it is shown that the closure of PFn is not necessarily WPFn. Nevertheless, there are two situations in which we can identify paths of matrices $A(\varepsilon)$ in PFn converging to any given matrix A in WPFn. One of

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these cases is that of normal matrices in WPFn, and the other is when the spectral radius is a simple eigenvalue.

In this note we address the following question: Can we determine a simple expression for the aforementioned matrices $A(\varepsilon)$ as a function of ε ? and in particular, can we write $A(\varepsilon)$ and its corresponding spectral characteristics (spectral radius and corresponding eigenvector) as a polynomial in ε of low degree (with matrix coefficients)? In other words, what can we say about a path of matrices $A(\varepsilon)$ with the strong P-F property that converges to a given matrix A possessing the P-F property as $\varepsilon \rightarrow 0$? We show that it is possible to have $A(\varepsilon)$ cubic, $\rho[A(\varepsilon)]$ quadratic and the corresponding positive P-F eigenvector linear in ε (where the constant term of the corresponding polynomials are the true characteristics of A). In particular, the result demonstrates that it is possible to construct a matrix satisfying the strong P-F property such that the matrix, its spectral radius and its positive P-F eigenvector approximate the unperturbed values with order $O(\varepsilon)$. For normal matrices in WPFn and those in WPFn for which the spectral radius is simple, we present approximating matrices that satisfy the strong P-F property and are polynomials of small degree in trigonometric functions of ε ; the corresponding positive P-F eigenvectors have a similar expansion whereas the corresponding spectral radii are linear in ε .

2. Polynomial representation of approximating sequences. To answer the questions posed in the introduction on the form of the sequence $A(\varepsilon)$ with the strong P-F property converging to A with the P-F property, we begin with the following result.

THEOREM 2.1. *Let A be an $n \times n$ real matrix with the P-F property and let v be a semipositive eigenvector of A corresponding to $\rho(A)$. Then, there exist $n \times n$ real matrices A_1 , A_2 and A_3 , an n -vector v_1 and scalars ρ_1 and ρ_2 such that for all sufficiently small positive scalars ε , the matrix*

$$(2.1) \quad A(\varepsilon) = A + \varepsilon A_1 + \varepsilon^2 A_2 + \varepsilon^3 A_3$$

has the strong P-F property, $\rho[A(\varepsilon)] = \rho(A) + \varepsilon \rho_1 + \varepsilon^2 \rho_2$ and a corresponding positive eigenvector of $A(\varepsilon)$ is $v(\varepsilon) = v + \varepsilon v_1$. Furthermore, if $\rho(A)$ is a simple, positive and strictly dominant eigenvalue of A , then $A_3 = 0$ and $\rho_2 = 0$.

Proof. The proof is constructive. Let v be a P-F eigenvector of A , that is, v is semi-positive and $Av = \rho(A)v$. Let P be an $n \times n$ nonsingular real matrix such that $P^{-1}AP = J(A)$ is in real Jordan canonical form. We may assume that v is the first column of P and that the first diagonal block of $P^{-1}AP$ corresponds to $\rho(A)$. Consider the vector w given by

$$(2.2) \quad w^T e_i = \begin{cases} 1 & \text{if } v^T e_i = 0 \\ 0 & \text{if } v^T e_i \neq 0, \end{cases}$$

where e_i ($i = 1, \dots, n$) denotes the i^{th} canonical vector of \mathbb{R}^n , i.e., the i^{th} entry of e_i is 1 while all the other entries are 0's. The vector w satisfies $w^T v = 0$ and its nonzero coordinates are all ones, in particular, $w = 0$ if and only if $v > 0$. For any $\varepsilon > 0$, let $P(\varepsilon) = P + \varepsilon w e_1^T$ and let $J(\varepsilon) = J(A) + \delta \varepsilon e_1 e_1^T$, where $\delta = 0$ if $\rho(A)$ is a simple positive and strictly dominant eigenvalue of A , or else $\delta = 1$. Note that for all scalars $\varepsilon > 0$, it is true that $\rho[J(\varepsilon)] = \rho(A) + \delta \varepsilon$ is a simple positive and strictly dominant eigenvalue of $J(\varepsilon)$ with a corresponding eigenvector e_1 . Furthermore, for a sufficiently small $\varepsilon > 0$, it holds that $P(\varepsilon)$ is nonsingular and for any such ε ,

the matrix $B(\varepsilon) = P(\varepsilon)J(\varepsilon)P(\varepsilon)^{-1}$ has $\rho[B(\varepsilon)] = \rho[J(\varepsilon)] = \rho(A) + \delta\varepsilon$ as a simple positive and strictly dominant eigenvalue of $B(\varepsilon)$ with a corresponding eigenvector $v(\varepsilon) = P(\varepsilon)e_1 = v + \varepsilon w > 0$. Therefore, $B(\varepsilon)$ has the strong P-F property. In order to build $A(\varepsilon)$ from $B(\varepsilon)$, we observe that $P(\varepsilon)^{-1}$ can be expressed explicitly by

$$P(\varepsilon)^{-1} = P^{-1} - \frac{\varepsilon P^{-1} w e_1^T P^{-1}}{1 + \varepsilon e_1^T P^{-1} w};$$

the above is easy to verify directly (in fact, it is an instance of the Sherman-Morrison-Woodbury formula; see, e.g., [6]). Letting $A(\varepsilon) := (1 + \varepsilon e_1^T P^{-1} w)B(\varepsilon)$, we have that $A(\varepsilon)$ has the strong P-F property. Furthermore,

$$\begin{aligned} A(\varepsilon) &= (P + \varepsilon w e_1^T)[J(A) + \delta \varepsilon e_1 e_1^T][(1 + \varepsilon e_1^T P^{-1} w)P^{-1} - \varepsilon P^{-1} w e_1^T P^{-1}] \\ &= (P + \varepsilon w e_1^T)[J(A) + \delta \varepsilon e_1 e_1^T]\{P^{-1} + \varepsilon[(e_1^T P^{-1} w)P^{-1} - P^{-1} w e_1^T P^{-1}]\}. \end{aligned}$$

Thus, $A(\varepsilon)$ has a representation $A_0 + \varepsilon A_1 + \varepsilon^2 A_2 + \varepsilon^3 A_3$ with $A_0 = P J(A) P^{-1} = A$, $A_3 = \delta w e_1^T [(e_1^T P^{-1} w)P^{-1} - P^{-1} w e_1^T P^{-1}]$ and corresponding A_1 and A_2 . Further,

$$\rho[A(\varepsilon)] = (1 + \varepsilon e_1^T P^{-1} w)[\rho(A) + \varepsilon \delta] = \rho(A) + \varepsilon \rho_1 + \varepsilon^2 \rho_2$$

with $\rho_1 = \delta + \rho(A)e_1^T P^{-1} w$ and $\rho_2 = \delta e_1^T P^{-1} w$, and $v(\varepsilon) = v + \varepsilon w$ is a corresponding positive eigenvector. In particular, if $\rho(A)$ is a simple, positive and strictly dominant eigenvalue of A , then $\delta = 0$ which implies $A_3 = 0$ and $\rho_2 = 0$. \square

THEOREM 2.2. *Let A be a normal $n \times n$ real matrix with the P-F property, and let v be its P-F eigenvector. Then, for all sufficiently small positive scalars ε , there exists an approximating sequence of normal matrices $A(\varepsilon)$ in PFn (and hence having the strong P-F property) having the form*

$$A(\varepsilon) = A + \varepsilon A_1 + \sum_{1 \leq j+k \leq 2} \sin^j \varepsilon (\cos \varepsilon - 1)^k A_{jk} + \varepsilon \sum_{1 \leq j+k \leq 2} \sin^j \varepsilon (\cos \varepsilon - 1)^k B_{jk}$$

where the matrices A_1 , A_{jk} , and B_{jk} are real $n \times n$ matrices, their spectral radius has the form $\rho[A(\varepsilon)] = \rho(A) + \varepsilon$, and the corresponding eigenvector is $v(\varepsilon) = (\cos \varepsilon)v + (\sin \varepsilon)u$, where $u^T v = 0$. Furthermore, if $\rho(A)$ is simple positive and strictly dominant eigenvalue, then $B_{jk} = 0$ and $\rho[A(\varepsilon)] = \rho(A)$, and if A has a positive P-F vector, then $A_{jk} = B_{jk} = 0$, and $v(\varepsilon) = v$.

Proof. This proof is also constructive. Let A be a normal $n \times n$ real matrix with the P-F property and let v be a P-F eigenvector of A . Then, $A = P M P^T$, where P is an $n \times n$ real orthogonal matrix and M is a direct sum of 1×1 real blocks or positive scalar multiples of 2×2 real orthogonal blocks; see, e.g., [7, Theorem 2.5.8]. We may assume that the first diagonal block of M is the 1×1 block $[\rho(A)]$ and that v is the first column of P . Note that v is in this case a unit vector and that it is both a right and a left P-F eigenvector of A . Let ε be any given nonnegative real number. We define a matrix Q_ε as follows: If v is a positive vector then let Q_ε denote the $n \times n$ identity matrix for all $\varepsilon \in [0, \infty)$ otherwise consider the vector w given by (2.2). Then, the vector $u := w/\|w\|$ is a semipositive vector of unit length which is orthogonal to v . For all $\varepsilon \in [0, \frac{\pi}{2})$ define the orthogonal matrix

$$(2.3) \quad Q_\varepsilon := I + (\cos \varepsilon - 1)(v v^T + u u^T) + \sin \varepsilon (u v^T - v u^T),$$

and then define the matrix

$$(2.4) \quad A(\varepsilon) := Q_\varepsilon P (M + \varepsilon \delta e_1 e_1^T) P^T Q_\varepsilon^T,$$

where $\delta = 0$ if $\rho(A)$ is a simple positive and strictly dominant eigenvalue of A or else $\delta = 1$. Observe that the spectral radius of the matrix $A(\varepsilon)$ (which is $\rho(A) + \varepsilon\delta$) is a simple positive and strictly dominant eigenvalue of $A(\varepsilon)$ for all $\varepsilon \in (0, \frac{\pi}{2})$ and that the vector $Q_\varepsilon v = (\cos \varepsilon)v + (\sin \varepsilon)u$ is a positive right and left P-F eigenvector of $A(\varepsilon)$ for all $\varepsilon \in (0, \frac{\pi}{2})$. Hence, $A(\varepsilon)$ is in PFn and thus $A(\varepsilon)$ has the strong P-F property. Moreover, it follows from (2.4) that $A(\varepsilon)$ is unitarily diagonalizable and therefore is normal. Taking into consideration the explicit form of Q_ε from (2.3), we can write the matrix $A(\varepsilon)$ as follows:

$$\begin{aligned} A(\varepsilon) &= Q_\varepsilon(PMP^T + \varepsilon\delta Pe_1 e_1^T P^T)Q_\varepsilon^T \\ &= Q_\varepsilon(A + \varepsilon\delta vv^T)Q_\varepsilon^T \\ &= A + \varepsilon vv^T + \sum_{1 \leq j+k \leq 2} \sin^j \varepsilon (\cos \varepsilon - 1)^k A_{jk} \\ &\quad + \varepsilon \delta \sum_{1 \leq j+k \leq 2} \sin^j \varepsilon (\cos \varepsilon - 1)^k B_{jk}, \end{aligned}$$

where A_{jk} and B_{jk} are real $n \times n$ matrices. Furthermore, it follows from (2.4) that $A(\varepsilon) \rightarrow A$ as $\varepsilon \rightarrow 0$ and that if v is a positive vector then $Q_\varepsilon = I$ and thus $A(\varepsilon) = A + \varepsilon vv^T$. \square

A normal $n \times n$ real matrix A has the P-F property if and only if A is in WPFn. Hence, Theorem 2.2 gives the form of a normal approximating sequence of matrices $A(\varepsilon)$ in PFn that converges to a given normal matrix A in WPFn as $\varepsilon \rightarrow 0$ (even though it is not true that WPFn is the closure of PFn; see [2]). However, if we consider matrices A in WPFn for which $\rho(A)$ is a simple eigenvalue then we obtain the next result.

THEOREM 2.3. *Let A be a matrix in WPFn such that $\rho(A)$ is a simple eigenvalue, and let u and v be the corresponding right and left P-F eigenvectors, respectively. Then, there is an approximating sequence of matrices $A(\varepsilon)$ in PFn of the form:*

$$A(\varepsilon) = \sum_{k,m} (\cos^k \varepsilon \sin^m \varepsilon) B_{km} + \sum_{k,m} (\varepsilon \cos^k \varepsilon \sin^m \varepsilon) C_{km},$$

where k and m are integers such that $0 \leq k \leq 6$ and $0 \leq m \leq 2$; B_{km} and C_{km} are real $n \times n$ matrices, their spectral radius have the form $\rho[A(\varepsilon)] = \rho(A) + \varepsilon$, and the corresponding P-F eigenvectors have the form

$$(2.5) \quad u(\varepsilon) = u + \sum_{k,m} (\cos^k \varepsilon \sin^m \varepsilon) \hat{B}_{km} u,$$

$$(2.6) \quad v(\varepsilon) = v + \sum_{k,m} (\cos^k \varepsilon \sin^m \varepsilon) \hat{B}_{km} v,$$

where $0 \leq k \leq 3$, $0 \leq m \leq 2$, and \hat{B}_{km} are real $n \times n$ matrices. Thus $A(\varepsilon) \rightarrow A$ as $\varepsilon \rightarrow 0$. Furthermore, if $\rho(A)$ is a strictly dominant eigenvalue then $C_{km} = 0$ for all k and m .

Proof. Consider a matrix A in WPFn for which $\rho(A)$ is a simple eigenvalue. Let $A = P[[\rho(A)] \oplus J_2]P^{-1}$ be the real Jordan decomposition of matrix A , where J_2 is the direct sum of all the real Jordan blocks that correspond to eigenvalues other than $\rho(A)$ and suppose that u and v are respectively the first column of P and the transpose of the first row of P^{-1} . Thus, u and v are, respectively, right and left eigenvectors of A corresponding to $\rho(A)$. Moreover, $u^T v = v^T u = 1 > 0$ since $P^{-1}P = I$.

Let a nonnegative scalar ε be given. We begin by finding an orthogonal matrix Q_ε that converges to the identity matrix as $\varepsilon \rightarrow 0$ and that maps the two semipositive vectors u and v simultaneously to a pair of positive vectors for all sufficiently small positive values of ε . Most of the proof that follows is dedicated to constructing Q_ε . The orthogonal matrix Q_ε will be defined as the product of three orthogonal matrices $Q_{(j,\varepsilon)}$ ($j = 1, 2, 3$) which are rotations.

Partition the set $\langle n \rangle$ by writing $\langle n \rangle = \alpha_1 \cup \alpha_2 \cup \alpha_3 \cup \alpha_4$ where $\alpha_1 = \{j \mid u_j = v_j = 0\}$, $\alpha_2 = \{j \mid u_j > 0 \text{ and } v_j = 0\}$, $\alpha_3 = \{j \mid u_j > 0 \text{ and } v_j > 0\}$, and $\alpha_4 = \{j \mid u_j = 0 \text{ and } v_j > 0\}$. Let k_j denote the cardinality of α_j for $j = 1, 2, 3, 4$, and note that $k_3 \neq 0$ because $u^T v > 0$. We may assume that the elements of α_1 are the first k_1 integers in $\langle n \rangle$, the elements of α_2 are the following k_2 integers in $\langle n \rangle$, the elements of α_3 are the following k_3 integers in $\langle n \rangle$, and the elements of α_4 are the last k_4 integers in $\langle n \rangle$, i.e., $\alpha_1 = \{1, 2, \dots, k_1\}$, $\alpha_2 = \{k_1 + 1, k_1 + 2, \dots, k_1 + k_2\}$, $\alpha_3 = \{k_1 + k_2 + 1, k_1 + k_2 + 2, \dots, k_1 + k_2 + k_3\}$, and $\alpha_4 = \{k_1 + k_2 + k_3 + 1, k_1 + k_2 + k_3 + 2, \dots, n\}$. Let w_1 denote the vector $e_{k_1 + k_2 + 1}$. If the cardinality of α_2 is zero, i.e., $k_2 = 0$ then let $Q_{(1,\varepsilon)} = I$ otherwise define the vector

$$w_4[\alpha_j] = \begin{cases} 0 & \text{if } j = 1, 3, 4 \\ \frac{1}{\|u[\alpha_2]\|} u[\alpha_2] & \text{if } j = 2 \end{cases}$$

and let

$$Q_{(1,\varepsilon)} = I + (\cos \varepsilon - 1)(w_1 w_1^T + w_4 w_4^T) - \sin \varepsilon (w_1 w_4^T - w_4 w_1^T)$$

where $\varepsilon \in [0, \delta_1]$ and δ_1 is a sufficiently small positive scalar. Similarly, if the cardinality of α_4 is zero, i.e., $k_4 = 0$ then let $Q_{(2,\varepsilon)} = I$ otherwise define the vector

$$w_3[\alpha_j] = \begin{cases} 0 & \text{if } j = 1, 2, 3 \\ \frac{1}{\|v[\alpha_4]\|} v[\alpha_4] & \text{if } j = 4 \end{cases}$$

and let

$$Q_{(2,\varepsilon)} = I + (\cos \varepsilon - 1)(w_1 w_1^T + w_3 w_3^T) - \sin \varepsilon (w_1 w_3^T - w_3 w_1^T)$$

where $\varepsilon \in [0, \delta_2]$ and δ_2 is a sufficiently small positive scalar. Furthermore, if the cardinality of α_1 is zero, i.e., $k_1 = 0$ then let $Q_{(3,\varepsilon)} = I$ otherwise define the vector $w_2 = (k_1)^{-1/2} \sum_{j=1}^{k_1} e_j$ and let

$$Q_{(3,\varepsilon)} = I + (\cos \varepsilon - 1)(w_1 w_1^T + w_2 w_2^T) - \sin \varepsilon (w_1 w_2^T - w_2 w_1^T)$$

where $\varepsilon \in [0, \delta_3]$ and δ_3 is any scalar in the open interval $(0, \frac{\pi}{2})$. Define the rotation $Q_\varepsilon := Q_{(3,\varepsilon)} Q_{(2,\varepsilon)} Q_{(1,\varepsilon)}$ for all $\varepsilon \in [0, \delta]$ where $\delta = \min\{\delta_1, \delta_2, \delta_3\}$. Thus,

$$\begin{aligned} Q_\varepsilon = & I + (\cos^3 \varepsilon - 1)w_1 w_1^T - (\sin \varepsilon)w_1 w_2^T - (\cos \varepsilon \sin \varepsilon)w_1 w_3^T - (\cos^2 \varepsilon \sin \varepsilon)w_1 w_4^T \\ & + (\cos^2 \varepsilon \sin \varepsilon)w_2 w_1^T + (\cos \varepsilon - 1)w_2 w_2^T - (\sin^2 \varepsilon)w_2 w_3^T - (\cos \varepsilon \sin^2 \varepsilon)w_2 w_4^T \\ & + (\cos \varepsilon \sin \varepsilon)w_3 w_1^T + (\cos \varepsilon - 1)w_3 w_3^T - (\sin^2 \varepsilon)w_3 w_4^T + (\sin \varepsilon)w_4 w_1^T \\ & + (\cos \varepsilon - 1)w_4 w_4^T. \end{aligned}$$

Define the approximating matrix $A(\varepsilon)$ as follows:

$$(2.7) \quad A(\varepsilon) := Q_\varepsilon P [[\rho(A) + \varepsilon] \oplus J_2] (Q_\varepsilon P)^{-1}$$

for all ε in $[0, \delta]$. The matrix $A(\varepsilon)$ is in PFn for all ε in $(0, \delta]$ and the right and left P-F eigenvectors of $A(\varepsilon)$ are $Q_\varepsilon u$ and $Q_\varepsilon v$ respectively, which have the form (2.5) and (2.6). Moreover, it is clear from the form of Q_ε and (2.7) that $A(\varepsilon)$ can be written as follows:

$$\begin{aligned} A(\varepsilon) &= Q_\varepsilon P [[\rho(A) + \varepsilon] \oplus J_2] P^{-1} Q_\varepsilon^T \\ &= \sum_{k,m} (\cos^k \varepsilon \sin^m \varepsilon) B_{km} + \sum_{k,m} (\varepsilon \cos^k \varepsilon \sin^m \varepsilon) C_{km} \end{aligned}$$

where k and m are integers such that $0 \leq k \leq 6$ and $0 \leq m \leq 2$; B_{km} and C_{km} are real $n \times n$ matrices; and $A(\varepsilon) \rightarrow A$ as $\varepsilon \rightarrow 0$. Furthermore, if $\rho(A)$ is a strictly dominant eigenvalue then $C_{km} = 0$ for all k and m . \square

REMARK 2.4. The approximating matrices in Theorems 2.2 and 2.3 can be written as power series in ε after replacing $\cos \varepsilon$ and $\sin \varepsilon$ with their corresponding Taylor series.

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