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may not have a nonnegative inverse**

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GENERALIZATIONS OF M -MATRICES WHICH MAY NOT HAVE A NONNEGATIVE INVERSE*

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Dedicated to Richard S. Varga on the occasion of his 80th birthday

Abstract. Generalizations of M -matrices are studied, including the new class of GM -matrices. The matrices studied are of the form $sI - B$ with B having some Perron-Frobenius property, but not necessarily being nonnegative. Results for these classes of matrices are shown, which are analogous to those known for M -matrices. Also, various splittings of a GM -matrix are studied along with conditions for their convergence.

Key words. Generalized M -matrices, Perron-Frobenius property.

AMS subject classifications. 15A48

1. Introduction. Nonnegative matrices possess the Perron-Frobenius property, i.e., each nonnegative matrix has a dominant positive eigenvalue that corresponds to a nonnegative eigenvector. In a recent paper [6], we studied matrices that are not necessarily nonnegative yet possess the Perron-Frobenius property. Closely related to this subject is the subject of M -matrices. A matrix $A \in \mathbb{R}^{n \times n}$ is called an M -matrix if it can be expressed as $A = sI - B$ where B is nonnegative and has a spectral radius $\rho(B) \leq s$.

In this paper, we study generalizations of M -matrices of the form $A = sI - B$ where B and B^T possess the Perron-Frobenius property and $\rho(B) \leq s$. We call such matrices GM -matrices. We also study other generalizations of this type and present some of their properties which are counterparts to those of M -matrices.

Recall that a matrix B is said to be *eventually nonnegative* (*eventually positive*) if $B^k \geq 0$ ($B^k > 0$, respectively) for all $k \geq p$ for some positive integer p .

Among the generalizations of M -matrices we study are matrices of the form $A = sI - B$ with $\rho(B) \leq s$ and B being an eventually nonnegative or an eventually positive matrix. Johnson and Tarazaga [15] termed the latter class, pseudo- M -matrices. Le and McDonald [16] studied the case where B is an irreducible eventually nonnegative matrix. We mention also other generalizations of M -matrices not considered in this paper; namely, where B leaves a cone invariant (see, e.g., [22], [24]) or for rectangular matrices; see, e.g., [20].

It is well-known that the inverse of a nonsingular M -matrix is nonnegative [1], [23] (and we prove an analogous result for GM -matrices in section 3). This property leads to the natural question: for which nonnegative matrices is the inverse an M -matrix? This question and related topics were extensively studied; see, e.g., [4], [5], [7]–[10], [12]–[14], [17]. In section 3, we study analogous questions, such as: for which matrices having the Perron-Frobenius property is the inverse a GM -matrix?

Another aspect we address (in section 4) is the study of splittings $A = M - N$ of a GM -matrix A and of conditions for their convergence.

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2. Notation and Preliminaries. We say that a real or complex matrix A is nonnegative (positive, nonpositive, negative, respectively) if it is entry-wise nonnegative (positive, nonpositive, negative, respectively) and we write $A \geq 0$ ($A > 0$, $A \leq 0$, $A < 0$, respectively). This notation and nomenclature is also used for vectors. If v is a nonzero and nonnegative column or row vector then we say v is *semipositive*. The spectral radius of a matrix A is denoted by $\rho(A)$. The spectrum of a matrix A is denoted by $\sigma(A)$. We call an eigenvalue of A a simple eigenvalue if its algebraic multiplicity in the characteristic polynomial is 1. We call an eigenvalue $\lambda \in \sigma(A)$ *dominant* if $|\lambda| = \rho(A)$. We call an eigenvalue $\lambda \in \sigma(A)$ *strictly dominant* if $|\lambda| > |\mu|$ for all $\mu \in \sigma(A)$, $\mu \neq \lambda$. The ordinary eigenspace of A for the eigenvalue λ is denoted by $E_\lambda(A)$. By definition, $E_\lambda(A) = \mathcal{N}(A - \lambda I)$, the null space of $A - \lambda I$. The nonzero vectors in $E_\lambda(A)$ are called ordinary eigenvectors of A corresponding to λ .

We say that a matrix $A \in \mathbb{R}^{n \times n}$ has the *Perron-Frobenius property* if the spectral radius is a positive eigenvalue that has an entry-wise nonnegative eigenvector. Also, we say that a matrix $A \in \mathbb{R}^{n \times n}$ has the *strong Perron-Frobenius property* if the spectral radius $\rho(A)$ is a simple positive eigenvalue that is strictly larger in modulus than any other eigenvalue and there is an entry-wise positive eigenvector corresponding to $\rho(A)$. By PF_n we denote the collection of matrices that are eventually positive. It turns out that $A \in PF_n$ if and only if A and A^T possess the strong Perron-Frobenius property; see, e.g., [15] and [18]. By WPF_n we denote the collection of matrices A for which both A and A^T possess the Perron-Frobenius property. The containments in the following statement are proper; see [6, Section 5].

$$PF_n \subset \{\text{nonnilpotent eventually nonnegative matrices}\} \subset WPF_n.$$

We say that $A \in \mathbb{C}^{1 \times 1}$ is reducible if $A = [0]$. We say that $A \in \mathbb{C}^{n \times n}$ ($n \geq 2$) is reducible if A is permutationally similar to $\begin{bmatrix} B & O \\ C & D \end{bmatrix}$ where B and D are square matrices. We say that a matrix $A \in \mathbb{C}^{n \times n}$ ($n \geq 1$) is irreducible if A is not reducible.

For an $n \times n$ matrix A , we define the graph $G(A)$ to be the graph with vertices $1, 2, \dots, n$ in which there is an edge (i, j) if and only if $a_{ij} \neq 0$. We say vertex i has access to vertex j if $i = j$ or else if there is a sequence of vertices (v_1, v_2, \dots, v_r) such that $v_1 = i$, $v_r = j$ and (v_i, v_{i+1}) is an edge in $G(A)$ for $i = 1, \dots, r-1$. If i has access to j and j has access to i then we say i and j communicate. Equivalence classes under the communication relation on the set of vertices of $G(A)$ are called the *classes* of A . By $A[\alpha]$ we denote the principal submatrix of $A \in \mathbb{R}^{n \times n}$ indexed by $\alpha \subseteq \{1, 2, \dots, n\}$. The graph $G(A[\alpha])$ is called a strong component of $G(A)$ whenever α is a class of A . We say that $G(A)$ is strongly connected whenever A has one class, or equivalently whenever A is irreducible. We call a class α basic if $\rho(A[\alpha]) = \rho(A)$. We call a class α initial if no vertex in any other class β has access to any vertex in α .

A matrix $A \in \mathbb{R}^{n \times n}$ is a Z -matrix if A can be expressed in the form $A = sI - B$ where s is a positive scalar and B is a nonnegative matrix. Moreover, if $A = sI - B$ is a Z -matrix such that $\rho(B) \leq s$, then we call A an M -matrix.

If $A \in \mathbb{R}^{n \times n}$ can be expressed as $A = sI - B$ where $B \in WPF_n$, then we call A

- a GZ -matrix.
- a GM -matrix if $0 < \rho(B) \leq s$.
- an EM -matrix if $0 < \rho(B) \leq s$ and B is eventually nonnegative.
- a pseudo- M -matrix if $0 < \rho(B) < s$ and $B \in PF_n$ [15].

When the inverse of a matrix C is a GM -matrix then we call C an inverse GM -matrix.

It follows directly from the definitions that every M -matrix is an EM -matrix,

that every EM -matrix is a GM -matrix, and that every pseudo- M -matrix is an EM -matrix. We show by examples below that the converses do not hold.

Furthermore, an M -matrix may not be a pseudo- M -matrix. Consider, for example, a reducible M -matrix. We illustrate the relations among the different sets of matrices in Figure 2.1.

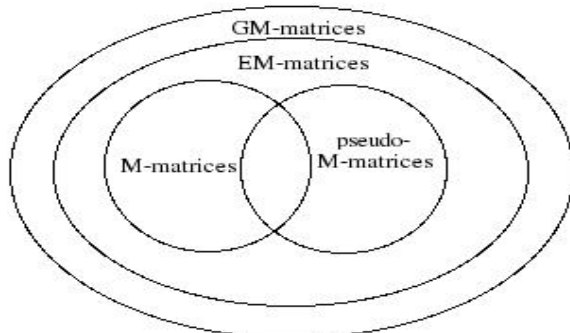


FIG. 2.1. This diagram summarizes the relations between the sets of various generalizations of M -matrices using the Perron-Frobenius property.

EXAMPLE 2.1.

$$\text{Let } A = sI - B \text{ where } B = \begin{bmatrix} 2 & 2 & 0 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 2 & 2 \\ 0 & 0 & -1 & 1 & 2 & 2 \end{bmatrix} \text{ and } s > 4.$$

Note that matrix B , which is taken from [3, Example 4.8], is a reducible nonnilpotent eventually nonnegative matrix with $\rho(B) = 4$. Hence, A is an EM -matrix. Since A is reducible, it follows that, for any positive scalar δ , we have $\delta I - A$ reducible and any power of $\delta I - A$ reducible. Hence, for any positive scalar δ , the matrix $\delta I - A$ is not eventually positive (i.e. $(\delta I - A) \notin PF_6$). And thus, A is not a pseudo- M -matrix. Moreover, A is not an M -matrix because A has positive off-diagonal entries.

EXAMPLE 2.2.

$$\text{Let } A = sI - B \text{ where } B = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & -1 & -1 \end{bmatrix} \text{ and } s > 2.$$

Note that $\rho(B) = 2$ is an eigenvalue having $[1 \ 1 \ 0 \ 0]^T$ a right and a left eigenvector. Hence, $B \in WPF_4$ and A is a GM -matrix. However, B is not eventually nonnegative because the lower right 2×2 block of B keeps on alternating signs. Moreover, for any positive scalar δ , the lower 2×2 block of $\delta I - A$ is the matrix $C = \begin{bmatrix} \delta - s - 1 & -1 \\ -1 & \delta - s - 1 \end{bmatrix}$. Note that for any positive integer k , the lower 2×2

block of $(\delta I - A)^k$ is the matrix C^k which is, using an induction argument, the matrix $\frac{1}{2} \begin{bmatrix} (\delta - s - 2)^k + (\delta - s)^k & (\delta - s - 2)^k - (\delta - s)^k \\ (\delta - s - 2)^k - (\delta - s)^k & (\delta - s - 2)^k + (\delta - s)^k \end{bmatrix}$. It is easy to see that for any choice of a positive scalar δ the matrix $\delta I - A$ is not eventually nonnegative because the (2,1)-entry of C^k are always negative for odd powers k .

3. Properties of GM-matrices. In this section, we generalize some results known for M -matrices to GM -matrices. For example, if A is a nonsingular M -matrix, then A^{-1} is nonnegative; see, e.g., [1], [23]. We show analogous results for GM - and pseudo- M -matrices. However, we show by an example that no analogous results for EM -matrices hold.

THEOREM 3.1. *Let A be a matrix in $\mathbb{R}^{n \times n}$ whose eigenvalues (when counted with multiplicity) are arranged in the following manner: $|\lambda_1| \geq |\lambda_2| \geq \dots \geq |\lambda_n|$. Then the following statements are equivalent:*

- (i) A is a nonsingular GM -matrix.
- (ii) $A^{-1} \in WPF_n$ and $0 < \lambda_n < Re(\lambda_i)$ for all $\lambda_i \neq \lambda_n$.

Proof. Suppose first that $A = sI - B$ is a nonsingular GM -matrix ($B \in WPF_n$ and $0 < \rho(B) < s$). Then, there are semipositive vectors v and w such that $Bv = \rho(B)v$ and $w^T B = \rho(B)w^T$. This implies that $A^{-1}v = (s - \rho(B))^{-1}v$ and that $w^T A^{-1} = (s - \rho(B))^{-1}w^T$. Thus, v and w are eigenvectors of A^{-1} and furthermore $\rho(A^{-1}) = |\lambda_n|^{-1} = (s - \rho(B))^{-1} > 0$. Therefore, $A^{-1} \in WPF_n$. Moreover, $|\lambda_n| = \lambda_n$, i.e., $Re(\lambda_n) > 0$ and $Im(\lambda_n) = 0$, otherwise, if $Re(\lambda_n) \leq 0$ then the eigenvalue $(s - \lambda_n) \in \sigma(B)$ satisfies $|s - \lambda_n| > |s - |\lambda_n|| = \rho(B)$, which is a contradiction. Or, if $Re(\lambda_n) > 0$ but $Im(\lambda_n) \neq 0$, then again, $|s - \lambda_n| > |s - |\lambda_n|| = \rho(B)$, which is a contradiction. Therefore, $|\lambda_n| = \lambda_n > 0$. Similarly, one could show that if $|\lambda_i| = \lambda_n$ for some $i \in \{1, \dots, n-1\}$ then $\lambda_i = \lambda_n > 0$. Furthermore, suppose that $\lambda_n \geq Re(\lambda_i)$ for some $\lambda_i \neq \lambda_n$, then $|\lambda_i| > \lambda_n$ (otherwise, $\lambda_i = \lambda_n$). If $Re(\lambda_i) = \lambda_n$, then $|\lambda_i| > Re(\lambda_i)$, therefore $|Im(\lambda_i)| > 0$. Thus,

$$|s - \lambda_i| = \sqrt{|s - Re(\lambda_i)|^2 + |Im(\lambda_i)|^2} > |s - Re(\lambda_i)| \geq |s - \lambda_n| = s - \lambda_n = \rho(B),$$

which is a contradiction because $s - \lambda_i$ is an eigenvalue of B . On the other hand, if $Re(\lambda_i) < \lambda_n$, then $s - Re(\lambda_i) > s - \lambda_n > 0$. Thus,

$$|s - \lambda_i| \geq |s - Re(\lambda_i)| > |s - \lambda_n| = s - \lambda_n = \rho(B),$$

which is again a contradiction because $s - \lambda_i$ is an eigenvalue of B . Therefore, $\lambda_n < Re(\lambda_i)$ for all $\lambda_i \neq \lambda_n$.

Conversely, suppose that $A^{-1} \in WPF_n$ and that $0 < \lambda_n < Re(\lambda_i)$ for all $\lambda_i \neq \lambda_n$. Then, there are semipositive vectors v and w such that $A^{-1}v = \rho(A^{-1})v = \lambda_n^{-1}v$ and $w^T A^{-1} = \rho(A^{-1})w^T = \lambda_n^{-1}w$. Note that for every λ_i such that $|\lambda_i| = \lambda_n$ we have $\lambda_i = \lambda_n$ (otherwise, $0 < \lambda_n < Re(\lambda_i) \leq |\lambda_i| = \lambda_n$, which is a contradiction). Moreover, the set of complex numbers $\{\lambda_i \in \sigma(A) : |\lambda_i| \neq \lambda_n\} = \sigma(A) \setminus \{\lambda_n\}$ lies completely in the set Ω defined by the intersection of the following two sets:

- The annulus $\{z : \lambda_n \leq |z| \leq |\lambda_1|\}$, and
- The (open) half-plane $\{z : Re(z) > Re(\lambda_n)\}$.

It is easy to see that there is a real number s large enough so that the circle centered at s of radius $s - \lambda_n$ surrounds all the complex numbers $\lambda_i \in \sigma(A)$, $\lambda_i \neq \lambda_n$ lying in Ω ; see Figure 3.1. For such an s , define the matrix $B_s := sI - A$. Then the eigenvalues of B_s are $s - \lambda_1, s - \lambda_2, \dots, s - \lambda_n$. Moreover, by our choice of s , we have the following:

$$|s - \lambda_i| < s - \lambda_n \quad \text{for all } \lambda_i \neq \lambda_n.$$

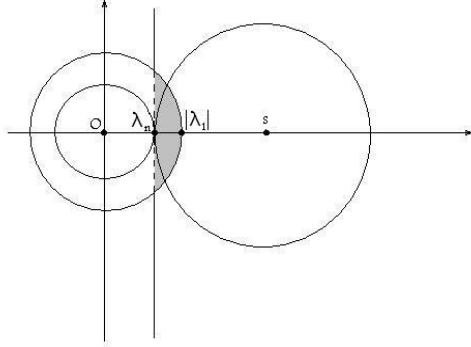


FIG. 3.1. The gray region represents the set Ω , which is the intersection of the open right half-plane determined by the vertical straight line passing through λ_n and the closed annulus centered at 0 with radii λ_n and $|\lambda_1|$.

Therefore, $0 < \rho(B_s) = s - \lambda_n < s$. Moreover, $B_s v = (s - \lambda_n)v$ and that $w^T B_s = (s - \lambda_n)w^T$. Thus, $B_s \in WPF_n$. And therefore, $A = sI - B_s$ is a nonsingular GM -matrix. \square

In [15, Theorem 8], Johnson and Tarazaga proved that if A is a pseudo- M -matrix, then $A^{-1} \in PF_n$. We extend this theorem by giving necessary and sufficient conditions for a matrix A to be a pseudo- M -matrix. The proof is very similar to that of Theorem 3.1, and thus, it is omitted.

THEOREM 3.2. *Let A be a matrix in $\mathbb{R}^{n \times n}$ whose eigenvalues (when counted with multiplicity) are arranged in the following manner: $|\lambda_1| \geq |\lambda_2| \geq \dots \geq |\lambda_n|$. Then the following statements are equivalent:*

- (i) A is a pseudo- M -matrix.
- (ii) A^{-1} exists, A^{-1} is eventually positive, and $0 < \lambda_n < \text{Re}(\lambda_i)$, for $i = 1, \dots, n - 1$.

REMARK 3.3. Since every M -matrix is a GM -matrix, it follows that condition (ii) in Theorem 3.1 can be used to check if a matrix is not an inverse M -matrix. In particular, if the real part of any eigenvalue is less than the minimum of all moduli of all eigenvalues then the given matrix is not an inverse M -matrix.

REMARK 3.4. The set WPF_n in Theorem 3.1 is not completely analogous to the set of nonnegative matrices. In other words, if we replace in Theorem 3.1 WPF_n by the set of nonnegative matrices and if we replace a GM -matrix by an M -matrix, then the statement of the theorem would not be correct. Similarly, in Theorem 3.2, PF_n is not completely analogous with the set of positive matrices. For example, we may find a nonnegative matrix whose inverse is a GM -matrix but not an M -matrix.

An example of the latter is the positive matrix $C = \frac{1}{36} \begin{bmatrix} 7 & 6 & 5 \\ 5 & 12 & 1 \\ 1 & 6 & 11 \end{bmatrix}$. Note that $C^{-1} = \begin{bmatrix} 7 & -2 & -3 \\ -3 & 4 & 1 \\ 1 & -2 & 3 \end{bmatrix} = sI - B$ where $s = 10$, $B = \begin{bmatrix} 3 & 2 & 3 \\ 3 & 6 & -1 \\ -1 & 2 & 7 \end{bmatrix} \in WPF_3$, and $\rho(B) = 8$. Hence, C^{-1} is a nonsingular GM -matrix. However, C^{-1} is not an M -matrix since it has some positive off-diagonal entries.

COROLLARY 3.5. *A matrix $C \in \mathbb{R}^{n \times n}$ is an inverse GM-matrix if and only if $C \in WPF_n$ and $Re(\lambda^{-1}) > \rho(C)^{-1}$ for all $\lambda \in \sigma(C)$, $\lambda \neq \rho(C)$.*

COROLLARY 3.6. *Every real eigenvalue of a nonsingular GM-matrix is positive.*

EXAMPLE 3.7. In this example, we show a nonsingular EM-matrix whose inverse is not eventually nonnegative. This implies that no result analogous to The-

orems 3.1 and 3.2 hold for this case. Let $A = \begin{bmatrix} 2 & -1 & -1 & 1 \\ -1 & 2 & 1 & -1 \\ -1 & -1 & 2 & -1 \\ -1 & -1 & -1 & 2 \end{bmatrix} = 3I -$

$\begin{bmatrix} 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} = 3I - B$. Then, $\rho(B) = 2$ and, using an induction argu-

ment, $B^k = \begin{bmatrix} 2^{k-1} & 2^{k-1} & 0 & 0 \\ 2^{k-1} & 2^{k-1} & 0 & 0 \\ k2^{k-1} & k2^{k-1} & 2^{k-1} & 2^{k-1} \\ k2^{k-1} & k2^{k-1} & 2^{k-1} & 2^{k-1} \end{bmatrix} \geq 0$ for all integers $k \geq 2$. Hence,

A is an EM-matrix. But, $A^{-1} = 3^{-2}(E + F)$ where $E = \begin{bmatrix} 6 & 3 & 0 & 0 \\ 3 & 6 & 0 & 0 \\ 9 & 9 & 6 & 3 \\ 9 & 9 & 3 & 6 \end{bmatrix}$ and

$F = \begin{bmatrix} 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Note that $EF = FE = 3F$ and $F^2 = 0$. Therefore, using

an induction argument, it is easy to check that $(A^{-1})^k = 3^{-2k}E^k + k3^{-k-1}F$. Hence, A^{-1} is not eventually nonnegative because the (1,4) and (2,3) entries are always negative.

It is well-known that a Z-matrix $A \in \mathbb{R}^{n \times n}$ is a nonsingular M-matrix if and only if A is positive stable; see, e.g., [1, p.137]. In the following proposition, we prove an analogous result between GZ-matrices and GM-matrices.

PROPOSITION 3.8. *A GZ-matrix $A \in \mathbb{R}^{n \times n}$ is a nonsingular GM-matrix if and only if A is positive stable.*

Proof. Let A be a GZ-matrix in $\mathbb{R}^{n \times n}$ with eigenvalues $|\lambda_1| \geq |\lambda_2| \geq \dots \geq |\lambda_n|$. If A is a nonsingular GM-matrix, then Theorem 3.1 implies that $0 < \lambda_n < Re(\lambda_i)$ for all $\lambda_i \neq \lambda_n$. Thus, $Re(\lambda_i) > 0$ for $i = 1, 2, \dots, n$. Hence, A is positive stable. Conversely, suppose that A is positive stable, then it follows that 0 is not an eigenvalue of A , which implies that A is nonsingular. Moreover, since A is a GZ-matrix we can decompose A in the following manner $A = sI - B$ where $B \in WPF_n$ and $s \geq 0$. If $s \leq \rho(B)$ then $(s - \rho(B))$ is a nonpositive eigenvalue of A , which contradicts the positive stability of A . Hence, $s > \rho(B)$, which shows that A is a nonsingular GM-matrix. \square

Another useful result is the following; see, e.g., [1, p.136].

THEOREM 3.9. *A Z-matrix $A \in \mathbb{R}^{n \times n}$ is a nonsingular M-matrix if and only if there is a positive vector x such that Ax is positive.*

In Theorem 3.11 below, we prove an analogous result for pseudo-M-matrices. The results in the following lemma are proved in [18, Theorem 2.6].

LEMMA 3.10. *If $B \in \mathbb{R}^{n \times n}$ has a left Perron-Frobenius eigenvector and $x = [x_1 \ \dots \ x_n]^T$ is any positive vector then either $\frac{\sum_{j=1}^n b_{ij}x_j}{x_i} = \rho(B)$ for all $i \in \{1, 2, \dots, n\}$*

or $\min_{i=1}^n \frac{\sum_{j=1}^n b_{ij}x_j}{x_i} \leq \rho(B) \leq \max_{i=1}^n \frac{\sum_{j=1}^n b_{ij}x_j}{x_i}$.

THEOREM 3.11. *If $A = sI - B$ where $B \in PF_n$, then the following are equivalent:*

- (i) A is a pseudo- M -matrix.
- (ii) There is a positive vector x such that Ax is positive.

Proof. Suppose $A = sI - B$ is a pseudo- M -matrix and let x be a right Perron-Frobenius eigenvector of B . Then, $Ax = (sI - B)x = (s - \rho(B))x$ is a positive vector. Conversely, suppose there is a positive vector x such that $Ax = (sI - B)x = sx - Bx$ is positive. Then, $\max_{i=1}^n \frac{\sum_{j=1}^n b_{ij}x_j}{x_i} < s$, and by Lemma 3.10, $\rho(B) \leq \max_{i=1}^n \frac{\sum_{j=1}^n b_{ij}x_j}{x_i}$. Hence, $\rho(B) < s$ which proves that A is a pseudo- M -matrix. \square

We next give a characterization of nonsingular GM -matrices which has the flavor of Theorem 3.9. We use the following result, which is [6, Lemma 10.7].

LEMMA 3.12. *For any semipositive vector v_1 and for any scalar $\epsilon > 0$, there is an orthogonal matrix Q such that $\|Q - I\|_2 < \epsilon$ and $Qv_1 > 0$.*

THEOREM 3.13. *If $A = sI - B$ is a GZ -matrix ($B \in WPF_n$), then the following are equivalent:*

- (i) A is a nonsingular GM -matrix.
- (ii) There is an orthogonal matrix Q such that Qx and $QA^T x$ are positive where x is a left Perron-Frobenius eigenvector of B .

Proof. Suppose $A = sI - B$ is a nonsingular GM -matrix and let x be a left Perron-Frobenius eigenvector of B . Then, by Lemma 3.12, there is an orthogonal matrix Q such that Qx is positive. Moreover, $QA^T x = Q(sI - B^T)x = (s - \rho(B))Qx$ is positive since A is nonsingular having $\rho(B) < s$. Hence, (i) \Rightarrow (ii). Conversely, suppose (ii) is true. Then, $QA^T x = Q(sI - B^T)x = (s - \rho(B))Qx$ is positive, and thus, $\rho(B) < s$. \square

We end this section with a result on the classes of an EM -matrix.

PROPOSITION 3.14. *Let $A = sI - B$ be an EM -matrix (B eventually nonnegative and $0 < \rho(B) \leq s$). If A is singular, then for every class α of B the following holds:*

1. $A[\alpha]$ is a singular irreducible EM -matrix if α is basic.
2. $A[\alpha]$ is a nonsingular irreducible EM -matrix if α is not basic.

Proof. If A is a singular EM -matrix and α is a class of B , then $A[\alpha] = sI - B[\alpha]$, where I is the identity matrix having the appropriate dimensions. If α is a basic class of B , then $B[\alpha]$ is an irreducible submatrix of B and $\rho(B[\alpha]) = \rho(B) > 0$. Since the eigenvalues of A are of the form $s - \mu$ where $\mu \in \sigma(B)$ and since A is singular, it follows that $\rho(B) = s$. Hence, $\rho(B[\alpha]) = s$ and $A[\alpha] = sI - B[\alpha]$ must be singular, as well. Moreover, since $B[\alpha]$ is irreducible, it follows that the graph $G(B[\alpha])$ is strongly connected. Note that the graph $G(A[\alpha]) = G(sI - B[\alpha])$ may differ from the graph $G(B[\alpha])$ only in having or missing some loops on some vertices. But this means that the graph $G(A[\alpha])$ is also strongly connected because adding or removing loops from vertices of a strongly connected graph does not affect strong connectedness. Hence, $A[\alpha]$ is irreducible. Moreover, if $\kappa = (\alpha_1, \dots, \alpha_m)$ is an ordered partition of $\{1, 2, \dots, n\}$ that gives the Frobenius normal form of B (see, e.g., [2]), then B_κ is block triangular and it is permutationally similar to B . Thus, B_κ is eventually nonnegative and so is each of its diagonal blocks. In particular, there is a diagonal block in B_κ which is permutationally similar to $B[\alpha]$ (because α is a class of B). Hence, $B[\alpha]$ is eventually nonnegative, which proves part 1. Similarly, if α is not a basic class of B , then part 2 holds. \square

4. Splittings and GM -matrices. In this section, we define various splittings of a GM -matrix, give sufficient conditions for convergence, and give several examples.

Recall that a splitting of a matrix A is an expression of the form $A = M - N$ where M is nonsingular. We begin by listing some preliminary definitions.

DEFINITION 4.1. *Let $A = M - N$ be a splitting. Then, such a splitting is called*

- *weak (or nonnegative) if $M^{-1}N \geq 0$.*
- *weak-regular if $M^{-1}N \geq 0$ and $M^{-1} \geq 0$ [19].*
- *regular if $M^{-1} \geq 0$ and $N \geq 0$ [23].*
- *M -splitting if M is an M -matrix and $N \geq 0$ [21].*
- *Perron-Frobenius splitting if $M^{-1}N$ possesses the Perron-Frobenius property [18].*

We list now the new splittings introduced in this paper. We begin first by defining the splitting having the Perron singular property, which is a splitting for an arbitrary nonsingular matrix. Then we proceed to define the splittings specific to nonsingular GM -matrices.

DEFINITION 4.2. *Let A be nonsingular. We say that the splitting $A = M - N$ has the Perron singular property if $\gamma M + (1 - \gamma)N$ is singular for some $\gamma \in \mathbb{C}$, $\gamma \neq 0$ and $M^{-1}N$ has the Perron-Frobenius property.*

Note that a splitting with the Perron singular property is, in particular, a Perron-Frobenius splitting.

DEFINITION 4.3. *Let $A = M - N$ be a splitting of a nonsingular GM -matrix $A = sI - B$ ($B \in WPF_n$ and $\rho(B) < s$). Then, such a splitting is called*

- *a G -regular splitting if M^{-1} and N are in WPF_n .*
- *a GM -splitting if M is a GM -matrix and $N \in WPF_n$.*
- *an overlapping splitting if $E_\lambda(M^{-1}N) \cap E_{\rho(B)}(B) \neq \{0\}$ with $|\lambda| = \rho(M^{-1}N)$.*
- *a commuting bounded splitting if M and N commute and $\rho(M) < s$.*

REMARK 4.4. A GM -splitting is a G -regular splitting but not conversely. For example, consider the GM -matrix $A = \text{diag}(1, 4, 4) = sI - B$ where $s = 5$ and $B = \text{diag}(4, 1, 1)$. An example of a G -regular splitting of A is the splitting $A = M - N$ where $M = \text{diag}(2, 32, -4)$ and $N = \text{diag}(1, 28, -8)$. Note that M^{-1} is in WPF_n yet, by Theorem 3.1, M is not a GM -matrix. Hence, this G -regular splitting is not a GM -splitting.

LEMMA 4.5. *Let $A = M - N$ be a splitting of a nonsingular matrix A . Then, the following are equivalent:*

- (i) *The splitting is convergent.*
- (ii) *$\min \{ \text{Re}(\lambda) \mid \lambda \in \sigma(NA^{-1}) \} > -\frac{1}{2}$.*
- (iii) *$\min \{ \text{Re}(\lambda) \mid \lambda \in \sigma(A^{-1}N) \} > -\frac{1}{2}$.*

Proof. We prove first the equivalence of (i) and (ii). Let $P = M^{-1}NA^{-1}M$. Thus, P and NA^{-1} are similar matrices, and therefore, they have the same eigenvalues with the same multiplicities. Moreover, the following relation between P and $M^{-1}N$ holds:

$$P = M^{-1}NA^{-1}M = M^{-1}N(M - N)^{-1}M = M^{-1}N(I - M^{-1}N)^{-1}.$$

Hence, the eigenvalues of NA^{-1} and $M^{-1}N$ are related as follows: $\mu \in \sigma(M^{-1}N)$ if and only if there is a unique $\lambda \in \sigma(NA^{-1})$ such that $\mu = \frac{\lambda}{1+\lambda}$. The splitting is convergent, i.e., $\rho(M^{-1}N) < 1$ if and only if for all $\mu \in \sigma(M^{-1}N)$, we have $|\mu| < 1$. That is, if for all $\lambda \in \sigma(NA^{-1})$, we have $\left| \frac{\lambda}{1+\lambda} \right| < 1$, or equivalently, $\frac{(\text{Re}(\lambda))^2 + (\text{Im}(\lambda))^2}{(1 + \text{Re}(\lambda))^2 + (\text{Im}(\lambda))^2} < 1$, which holds only whenever $2\text{Re}(\lambda) + 1 > 0$, or whenever (ii) is true. As for the equivalence of (i) and (iii), it follows similarly by noting the

following relation between $A^{-1}N$ and $M^{-1}N$:

$$A^{-1}N = (M - N)^{-1}N = (I - M^{-1}N)^{-1}M^{-1}N. \quad \square$$

COROLLARY 4.6. *Let $A = M - N$ be a splitting of a nonsingular matrix A . If $A^{-1}N$ or NA^{-1} is an inverse GM-matrix, then the splitting is convergent.*

Proof. Let P denote $A^{-1}N$ or NA^{-1} . If P is an inverse GM-matrix then, by Corollary 3.5, $Re(\lambda^{-1}) > (\rho(P))^{-1} > 0$ for all $\lambda \in \sigma(P)$, $\lambda \neq \rho(P)$. This implies that $Re(\lambda) = |\lambda|^2 Re(\lambda^{-1}) > -\frac{1}{2}$ for all $\lambda \in \sigma(P)$, $\lambda \neq \rho(P)$. Thus, $Re(\lambda) > -\frac{1}{2}$ for all $\lambda \in \sigma(P)$, which is equivalent to condition (ii) of Lemma 4.5 if $P = NA^{-1}$, or equivalent to condition (iii) of Lemma 4.5 if $P = A^{-1}N$. Hence, the given splitting is convergent. \square

The following lemma is Theorem 2.3 of [18] and Lemma 5.1 of [6].

LEMMA 4.7. *Let A be a nonnilpotent eventually nonnegative matrix. Then, both A and A^T possess the Perron-Frobenius property, i.e., $A \in WPF_n$.*

THEOREM 4.8. *If $A = M - N$ is a splitting having the Perron singular property, then any of the following conditions is sufficient for convergence:*

- (A1) $A^{-1}N$ is eventually positive.
- (A2) $A^{-1}N$ is eventually nonnegative.
- (A3) $A^{-1}N \in WPF_n$.
- (A4) $A^{-1}N$ has a simple positive and strictly dominant eigenvalue with a positive spectral projector of rank 1.
- (A5) $A^{-1}N$ has a basic and an initial class α such that $(A^{-1}N)[\alpha]$ has a right Perron-Frobenius eigenvector.

Proof. We prove first (A2) \Rightarrow (A3) \Rightarrow convergence of the given splitting. Suppose that $A^{-1}N$ is eventually nonnegative. Since $A = M - N$ is a splitting having the Perron singular property, it follows that there is a nonzero complex scalar γ such that $\gamma M + (1 - \gamma)N$ is singular. Hence, $\gamma A + N$ is singular $\Leftrightarrow \det(\gamma A + N) = 0 \Leftrightarrow \det(-\gamma I - A^{-1}N) = 0$. In other words, $-\gamma$ is a nonzero eigenvalue of $A^{-1}N$, i.e., it is not nilpotent. By Lemma 4.7, $A^{-1}N$ and its transpose possess the Perron-Frobenius property, i.e., $A^{-1}N \in WPF_n$. And thus, the given splitting converges by Lemma 4.11. As for the rest of the sufficient conditions, we outline the proof using the following diagram:

$$\begin{array}{ccccccc} (A1) & \Rightarrow & (A2) & \Rightarrow & (A3) & \Rightarrow & \text{convergence} \\ & & \Downarrow & & \Uparrow & & \\ & & (A4) & & (A5) & & \end{array}$$

The equivalency and implications in the above diagram follow from the results in [6] on eventually positive matrices, eventually nonnegative matrices, and matrices in WPF_n . \square

REMARK 4.9. Recall that a regular splitting $A = M - N$ of a monotone matrix (i.e., when $A^{-1} \geq 0$) is convergent [23]. Thus, Theorem 4.8 is a generalization of this situation since we do not require that A^{-1} nor N , nor their product $A^{-1}N$ be nonnegative.

EXAMPLE 4.10. Let $A = \begin{bmatrix} 7 & -2 & -3 \\ -3 & 4 & 1 \\ 1 & -2 & 3 \end{bmatrix}$ and consider the splitting $A = M - N$, where $M = \frac{1}{4} \begin{bmatrix} 29 & -6 & -11 \\ -11 & 18 & 5 \\ 5 & -6 & 13 \end{bmatrix}$ and $N = \frac{1}{4} \begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \\ 1 & 2 & 1 \end{bmatrix}$. For $\gamma = -\frac{1}{2}$

the matrix $\gamma M + (1 - \gamma)N = \frac{1}{4} \begin{bmatrix} -13 & 6 & 7 \\ 7 & -6 & -1 \\ -1 & 6 & -5 \end{bmatrix}$ is singular. Moreover, $M^{-1}N =$

$\frac{1}{12} \begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \\ 1 & 2 & 1 \end{bmatrix}$ is a positive matrix and thus it possesses the Perron-Frobenius property. Hence, this splitting is a splitting with the Perron singular property. Since

$A^{-1}N = \frac{1}{8} \begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \\ 1 & 2 & 1 \end{bmatrix}$ is a positive matrix (and hence eventually positive), it follows from Theorem 4.8 that this splitting is convergent. In fact, $\rho(M^{-1}N) = \frac{1}{3} < 1$.

The following lemma is part of Theorem 3.1 of [18].

LEMMA 4.11. *Let $A = M - N$ be a Perron-Frobenius splitting of a nonsingular matrix $A \in \mathbb{R}^{n \times n}$. Then, the following are equivalent:*

- (i) $\rho(M^{-1}N) < 1$.
- (ii) $A^{-1}N$ possesses the Perron-Frobenius property.
- (iii) $\rho(M^{-1}N) = \frac{\rho(A^{-1}N)}{1 + \rho(A^{-1}N)}$.

COROLLARY 4.12. *Let $A = M - N$ be a splitting of a nonsingular matrix A such that N is nonsingular and $N^{-1}M$ is a nonsingular GM-matrix. Then, the following are equivalent:*

- (i) The splitting $A = M - N$ is convergent.
- (ii) $A^{-1}N$ possesses the Perron-Frobenius property.
- (iii) $\rho(M^{-1}N) = \frac{\rho(A^{-1}N)}{1 + \rho(A^{-1}N)}$.

Proof. Since $N^{-1}M$ is a nonsingular GM-matrix, it follows that $(N^{-1}M)^{-1} = M^{-1}N \in \text{WPF}_n$. Hence, $M^{-1}N$ satisfies the Perron-Frobenius property, which implies that the splitting $A = M - N$ is a Perron-Frobenius splitting and the equivalence of the statements in the corollary follows from Lemma 4.11. \square

PROPOSITION 4.13. *If $A = sI - B$ is a GM-matrix and the splitting $A = M - N$ is an overlapping splitting (for which $E_\lambda(M^{-1}N) \cap E_{\rho(B)}(B) \neq \{0\}$ and $|\lambda| = \rho(M^{-1}N)$), then such a splitting is convergent if and only if $\exists \eta = \frac{s - \rho(B)}{1 - \lambda} \in \sigma(M)$ such that $\text{Re}(\eta) > \frac{s - \rho(B)}{2}$.*

Proof. Note first that if $A = sI - B = M - N$ is an overlapping splitting then we can pick a semipositive vector $v \in E_{\rho(B)}(B) \cap E_\lambda(M^{-1}N)$. And for this vector, we have:

$$\begin{aligned} (sI - B)v &= Av = (M - N)v = M^{-1}(I - M^{-1}N)v \\ \Leftrightarrow (s - \rho(B))Mv &= (I - M^{-1}N)v = (1 - \lambda)v \\ \Leftrightarrow Mv &= \frac{(1 - \lambda)}{(s - \rho(B))}v \\ \Rightarrow \exists \eta \in \sigma(M) \ni \eta &= \frac{(1 - \lambda)}{(s - \rho(B))} \\ \Leftrightarrow \exists \eta \in \sigma(M) \ni \lambda &= \frac{\eta - (s - \rho(B))}{\eta}. \end{aligned}$$

Hence, if $A = M - N$ is an overlapping splitting then there is an eigenvalue $\eta \in \sigma(M)$ such that $\lambda = \frac{\eta - (s - \rho(B))}{\eta}$. Therefore, an overlapping splitting is convergent, i.e., $\rho(M^{-1}N) = |\lambda| < 1$, when $|\eta - (s - \rho(B))| < |\eta|$ for some $\eta \in \sigma(M)$, or equivalently whenever η lies in the right-half plane determined by the perpendicular bisector of

the segment on the real axis whose endpoints are 0 and $(s - \rho(B))$, i.e., whenever $\operatorname{Re}(\eta) > \frac{s - \rho(B)}{2}$. \square

COROLLARY 4.14. *Let $A = M - N$ be an overlapping splitting of a nonsingular GM -matrix A and suppose that $M^{-1}N \in WPF_n$. If $\frac{s - \rho(B)}{1 - \rho(M^{-1}N)} \in \sigma(M)$ then $\rho(M^{-1}N) < 1$, i.e. the splitting is convergent.*

EXAMPLE 4.15. Let A be as in Example 4.10. Then, A is a nonsingular GM -matrix. In fact, $A = sI - B$, where $s = 10$, $B = \begin{bmatrix} 3 & 2 & 3 \\ 3 & 6 & -1 \\ -1 & 2 & 7 \end{bmatrix} \in WPF_3$,

and $\rho(B) = 8$. An overlapping splitting of the matrix A is $A = M - N$ where $M = \frac{1}{8} \begin{bmatrix} 55 & -18 & -25 \\ -25 & 30 & 7 \\ 7 & -18 & 23 \end{bmatrix}$ and $N = \frac{1}{8} \begin{bmatrix} -1 & -2 & -1 \\ -1 & -2 & -1 \\ -1 & -2 & -1 \end{bmatrix}$. Note that $M^{-1}N =$

$\frac{2}{3} \begin{bmatrix} -1 & -2 & -1 \\ -1 & -2 & -1 \\ -1 & -2 & -1 \end{bmatrix}$ and that for $\lambda = -\frac{1}{3} \in \sigma(M^{-1}N)$ we have $|\lambda| = \frac{1}{3} = \rho(M^{-1}N)$

and $E_\lambda(M^{-1}N) = E_{\rho(B)}(B) = \operatorname{Span}\{[1 \ 1 \ 1]^T\}$. Hence, this overlapping splitting is convergent. Proposition 4.13 predicts the existence of an eigenvalue η of M such that $\eta = \frac{s - \rho(B)}{1 - \lambda} = \frac{10 - 8}{1 - (-1/3)} = \frac{3}{2}$ and $\operatorname{Re}(\eta) = \frac{3}{2} > \frac{s - \rho(B)}{2} = 1$. If we look at the spectrum of M we see that $\frac{3}{2} \in \sigma(M) = \{\frac{3}{2}, 6\}$ just as predicted by Proposi-

tion 4.13. On the other hand, if $A = M - N$ where $M = \frac{1}{4} \begin{bmatrix} 27 & -10 & -13 \\ -13 & 14 & 3 \\ 3 & -10 & 11 \end{bmatrix}$

and $N = \frac{1}{4} \begin{bmatrix} -1 & -2 & -1 \\ -1 & -2 & -1 \\ -1 & -2 & -1 \end{bmatrix}$ then $M^{-1}N = N = \frac{1}{4} \begin{bmatrix} -1 & -2 & -1 \\ -1 & -2 & -1 \\ -1 & -2 & -1 \end{bmatrix}$ and for

$\lambda = -1 \in \sigma(M^{-1}N)$ we have $|\lambda| = 1 = \rho(M^{-1}N)$ and $E_\lambda(M^{-1}N) = E_{\rho(B)}(B) = \operatorname{Span}\{[1 \ 1 \ 1]^T\}$. Hence, the latter splitting is an overlapping splitting but it does not converge. Proposition 4.13 predicts that for all $\eta \in \sigma(M)$ either $\eta \neq \frac{s - \rho(B)}{1 - \lambda} = \frac{10 - 8}{1 - (-1)} = 1$ or $\operatorname{Re}(\eta) \leq \frac{s - \rho(B)}{2} = 1$, which is true about the spectrum of M since $\sigma(M) = \{1, 6\}$.

THEOREM 4.16. *A GM -matrix $A = sI - B$ having a commuting bounded splitting $A = M - N$ induces a splitting of B of the form $B = M' - N'$ where $M' = \frac{1}{\omega}(sI - M)$, $\omega \in \mathbb{R}$ and $\omega \neq 0$. Moreover, if $|\omega| < \min \left\{ \frac{s}{\rho(M')}, \frac{1}{2\rho(M')} \right\}$, then the commuting bounded splitting is convergent.*

Proof. Suppose that the GM -matrix $A = sI - B$ has a commuting bounded splitting $A = M - N$ and let $M' = \frac{1}{\omega}(sI - M)$ and $N' = A - sI + M'$ for some $\omega \in \mathbb{R}$ and $\omega \neq 0$. Then, M' is nonsingular because $\rho(M) < s$ and $M = sI - \omega M'$. Moreover, we can write $A = (sI - \omega M') - ((1 - \omega)M' - N')$. Note that the iteration matrix of the commuting bounded splitting of A is $M^{-1}N = (sI - \omega M')^{-1}((1 - \omega)M' - N')$. Since M and N commute, so do M' and N' . Furthermore, there is a single unitary matrix U that produces the Schur decomposition (see, e.g., [11, p.81]) of both M' and N' . Hence, an eigenvalue of $M^{-1}N$ would have the form $(s - \omega\lambda)^{-1}((1 - \omega)\lambda - \mu)$ where $\lambda \in \sigma(M')$ and $\mu \in \sigma(N')$. But, since M' and N' are simultaneously Schur decomposable, it follows that the same unitary matrix that produces the Schur decomposition of M' and N' also produces the Schur decomposition of B and A . Therefore, $\lambda - \mu$ is an

eigenvalue of B which does not exceed s in modulus (since A is a GM -matrix). Thus,

$$|(s - \omega\lambda)^{-1}((1 - \omega)\lambda - \mu)| = \frac{|(\lambda - \mu) - \omega\lambda|}{|s - \omega\lambda|} \leq \frac{|\lambda - \mu| + |\omega||\lambda|}{|s - \omega\lambda|} \leq \frac{s + |\omega|\rho(M')}{|s - \omega\lambda|}$$

Moreover, if we choose $|\omega| < \frac{s}{\rho(M')}$, then $s - |\omega|\rho(M') > 0$. Hence,

$$|(s - \omega\lambda)^{-1}((1 - \omega)\lambda - \mu)| \leq \frac{s + |\omega|\rho(M')}{|s - \omega\lambda|} \leq \frac{s + |\omega|\rho(M')}{s - |\omega|\rho(M')}.$$

Therefore, if

$$(4.1) \quad \frac{s + |\omega|\rho(M')}{s - |\omega|\rho(M')} < 1$$

then the splitting $A = M - N$ is convergent. But (4.1) is equivalent to $|\omega| < \frac{1}{2\rho(M')}$.

Hence, if $|\omega| < \min \left\{ \frac{s}{\rho(M')}, \frac{1}{2\rho(M')} \right\}$, then the splitting $A = M - N$ is convergent. \square

EXAMPLE 4.17. Let $A = \frac{1}{40} \begin{bmatrix} 3 & 1 & 2 \\ 1 & 5 & 0 \\ 1 & 1 & 4 \end{bmatrix}$. Then, $A = sI - B$ is a GM -matrix,

where $s = 1$, $B = \frac{1}{40} \begin{bmatrix} 37 & -1 & -2 \\ -1 & 35 & 0 \\ -1 & -1 & 36 \end{bmatrix} \in WPF_3$, and $\rho(B) = 0.95$. A commut-

ing bounded splitting of A is $A = M - N$, where $M = \frac{1}{80} \begin{bmatrix} 73 & 2 & 1 \\ 1 & 74 & 1 \\ 1 & 2 & 73 \end{bmatrix}$ and

$N = \frac{1}{80} \begin{bmatrix} 67 & 0 & -3 \\ -1 & 64 & 1 \\ -1 & 0 & 65 \end{bmatrix}$. Note that $\rho(M) = 0.95 < 1 = s$ and that $MN = NM =$

$\frac{1}{1600} \begin{bmatrix} 1222 & 32 & -38 \\ -2 & 1184 & 34 \\ -2 & 32 & 1186 \end{bmatrix}$. Furthermore, let $\omega = 5$ and let $M' = \frac{1}{\omega}(sI - M) =$

$\frac{1}{400} \begin{bmatrix} 7 & -2 & -1 \\ -1 & 6 & -1 \\ -1 & -2 & 7 \end{bmatrix}$. Then, $\rho(M') = 0.02$ making $|\omega| = 5 < \min \left\{ \frac{s}{\rho(M')}, \frac{1}{2\rho(M')} \right\} =$

$\min \left\{ \frac{1}{0.02}, \frac{1}{2(0.02)} \right\} = 12.5$. Hence, by Theorem 4.16, the splitting $A = M - N$ is convergent. In fact, $\rho(M^{-1}N) \approx 0.9444 < 1$.

THEOREM 4.18. *If $A = M - N$ is a splitting of a GM -matrix A , then any Type I condition (listed below) implies that such a splitting is a G -regular splitting. Moreover, if the splitting $A = M - N$ is a G -regular splitting that satisfies one of Type II conditions (listed below), then any one of Type III conditions (listed below) is sufficient for convergence.*

Type I Conditions

- (D1) M^{-1} and N are eventually positive.
- (D2) M^{-1} and N are eventually nonnegative with N not being nilpotent.
- (D3) Each of M^{-1} and N has a simple positive and strictly dominant eigenvalue with a positive spectral projector of rank 1.

(D4) Each of M^{-1} and N has two classes α and β , not necessarily distinct, such that:

- (i) α is basic, initial, and $X[\alpha]$ has a right Perron-Frobenius eigenvector.
- (ii) β is basic, final, and $X[\beta]$ has a left Perron-Frobenius eigenvector.

Type II Conditions

- (E1) $M^{-1}N$ is eventually positive.
- (E2) $M^{-1}N$ is eventually nonnegative but not nilpotent.
- (E3) $M^{-1}N \in WPF_n$.
- (E4) $M^{-1}N$ has a simple positive and strictly dominant eigenvalue with a positive spectral projector of rank 1.
- (E5) $M^{-1}N$ has a basic and an initial class α such that $(M^{-1}N)[\alpha]$ has a right Perron-Frobenius eigenvector.

Type III Conditions

- (F1) $A^{-1}N$ is eventually positive.
- (F2) $A^{-1}N$ is eventually nonnegative but not nilpotent.
- (F3) $A^{-1}N \in WPF_n$.
- (F4) $A^{-1}N$ has a simple positive and strictly dominant eigenvalue with a positive spectral projector of rank 1.
- (F5) $A^{-1}N$ has a basic and an initial class α such that $(A^{-1}N)[\alpha]$ has a right Perron-Frobenius eigenvector.

Proof. We prove the theorem for the following Type I, Type II, and Type III conditions, respectively: (D1), (E1), and (F1), and then we outline the rest of the proof. Suppose that the splitting $A = M - N$ satisfies condition (D1). Then, (D1) is true if and only if M^{-1} and N are in $PF_n \subset WPF_n$. Hence, the splitting $A = M - N$ is a G -regular splitting. Moreover, suppose that $A = M - N$ is a G -regular splitting and that (E1) is true. Then, $M^{-1}N \in PF_n$ and thus $M^{-1}N$ has the Perron-Frobenius property. In particular, the given G -regular splitting becomes a Perron-Frobenius splitting. If (F1) is true then $A^{-1}N \in PF_n$ and thus $A^{-1}N$ possesses the Perron-Frobenius property. Hence, by Lemma 4.11, the G -regular splitting converges. With regards to the remaining conditions, we use the following diagrams to outline the proof:

$$\begin{array}{ccc}
 (D1) & \Rightarrow & (D2) \Rightarrow M^{-1}, N \in WPF_n \Leftrightarrow A = M - N \text{ is a } G\text{-regular splitting} \\
 \Downarrow & & \uparrow \\
 (D3) & & (D4)
 \end{array}$$

$$\begin{array}{ccc}
 & & (E5) \\
 & & \Downarrow \\
 (E1) & \Rightarrow & (E2) \Rightarrow (E3) \Rightarrow M^{-1}N \text{ has the Perron-Frobenius property} \\
 \Downarrow & & \Downarrow \\
 (E4) & & A = M - N \text{ is a Perron-Frobenius splitting} \\
 & & \text{as well as a } G\text{-regular splitting}
 \end{array}$$

$$\begin{array}{ccc}
 & & (F5) \\
 & & \Downarrow \\
 (F1) & \Rightarrow & (F2) \Rightarrow (F3) \Rightarrow A^{-1}N \text{ has the Perron-Frobenius property} \\
 \Downarrow & & \Downarrow \\
 (F4) & & \text{The splitting converges by Lemma 4.11}
 \end{array}$$

All the above implications and equivalencies follow from the results in [6] on eventually nonnegative matrices, eventually positive matrices, and matrices in WPF_n . \square

EXAMPLE 4.19. Let $A = \begin{bmatrix} 8 & -3 & -4 \\ -4 & 4 & 0 \\ 0 & -3 & 3 \end{bmatrix} = 8I - B = 8I - \begin{bmatrix} 0 & 3 & 4 \\ 4 & 4 & 0 \\ 0 & 3 & 5 \end{bmatrix}$.

Then, A is a nonsingular GM -matrix and $\rho(B) = \frac{1}{2}(7 + \sqrt{73}) \approx 7.7720 < 8$. Con-

sider the splitting $A = M - N$ where $M = \begin{bmatrix} 7 & -2 & -3 \\ -3 & 4 & 1 \\ 1 & -2 & 3 \end{bmatrix}$ (a nonsingular GM -

matrix from the previous examples) and $N = \begin{bmatrix} -1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \in WPF_3$. Thus,

this splitting of A is a GM -splitting, and hence, a G -regular splitting. Note that $M^{-1}N = \frac{1}{36} \begin{bmatrix} 4 & 12 & 13 \\ 8 & 6 & 17 \\ 16 & 12 & 7 \end{bmatrix}$ is an eventually positive matrix, a Type II condition

in Theorem 4.18. Moreover, $A^{-1}N = \frac{1}{12} \begin{bmatrix} 25 & 28 & 33 \\ 28 & 28 & 36 \\ 32 & 32 & 36 \end{bmatrix}$ is an eventually positive matrix, a Type III condition in Theorem 4.18. Hence, Theorem 4.18 predicts the convergence of this G -regular splitting. In fact, $\rho(M^{-1}N) \approx 0.8859 < 1$.

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