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Comparison theorems for the convergence factor of iterative methods for singular matrices

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Abstract

Iterative methods for the solution of consistent singular systems of linear equations are governed by the convergence factor of the iteration matrix *T*, i.e., by the quantity $\gamma(T) = \max\{|\lambda|, \lambda \in \sigma(T), \lambda \neq 1\}$, where $\sigma(T)$ is the spectrum of *T*. Theorems are presented comparing the convergence factor of two iterative methods. The comparison is based on the relationship between the matrices of the splittings. A cone other than the usual nonnegative hyperoctant is used to define the order used in this comparison. Although this cone is based on the (unknown) projection onto the null-space of a matrix, the characterization provided in the paper allows, in specific instances, the cone to be readily computable. © 2000 Elsevier Science Inc. All rights reserved.

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1. Introduction

The solution of $n \times n$ singular linear systems of equations of the form

$$Ax = b \tag{1}$$

is of great importance in many applications. In particular, when

$$A = I - B, \quad B^{\mathrm{T}}e = e, \quad e^{\mathrm{T}} = [1, 1, \dots, 1], \tag{2}$$

where B is the column stochastic matrix representing a Markov chain, and the solution of (1), for $b = 0$, is the stationary probability distribution of the Markov chain (normalized so that $x^{\mathrm{T}}e = 1$); see, e.g., [4,33]. In this case, $\rho(B) = 1$, where $\rho(B)$ denotes the spectral radius of B.

Iterative methods for the solution of (1) based on splittings of the form $A = M - N$, where M is nonsingular, have been successfully used for this problem; see, e.g., [1,2,15,19,27]. These methods include point and block versions of the classical Jacobi, Gauss–Seidel, and SOR methods [4,33,38] and can be written as the following iteration, starting from an initial vector $x_{(0)}$:

$$x_{(k+1)} = Tx_{(k)} + c, \quad c = M^{-1}b. \tag{3}$$

The matrix $T = M^{-1}N$ is called the iteration matrix, and it is generally assumed to be nonnegative (denoted $T \geqslant O$), e.g., when the splittings are weak regular [4], i.e., $M^{-1} \geqslant O$ and $M^{-1}N \geqslant O$. A regular splitting is such that $M^{-1} \geqslant O$ and $N \geqslant O$ [38]. A weak splitting is such that $M^{-1}N \geqslant O$ [18] (some authors call these splittings nonnegative splittings; see, e.g., [9,40]). Since $A = M(I - T)$, a singular matrix A implies that 1 is an eigenvalue of T, and $\rho(T) = 1$ is implied in the case of stochastic matrices such as in the case of Markov chains.

The rate of convergence of these iterative methods is governed by the quantity $\gamma(T) = \max\{|\lambda|, \lambda \in \sigma(T), \lambda \neq 1\}$, where $\sigma(T)$ is the spectrum of T. When $\gamma(T) = 1$ convergence is not guaranteed. When $\gamma(T) < 1$ and $\mathrm{ind}(I - T) = 1$ there is convergence; see, e.g., [4,35], and Section 2. We call the quantity $\gamma(T)$ the *convergence factor* of the iterative method (3). The quantity $\mathrm{ind}(I - T)$ is the maximal size of the Jordan blocks corresponding to the eigenvalue 1 of T; see, e.g., [4].

In the case of nonsingular A, the quantity governing the rate of convergence of the iterative methods is $\rho(T)$. There exists a rich literature comparing two splittings of the same matrix; see, e.g., [9,10,13,16,18,23,39,40]. For example, the following result can be found in [38].

Theorem 1.1. *Let A be a nonsingular matrix with $A^{-1} \geqslant O$ and let $A = M_1 - N_1 = M_2 - N_2$ be two regular splittings. If*

$$N_1 \leqslant N_2, \tag{4}$$

then $\rho(M_1^{-1}N_1) \leqslant \rho(M_2^{-1}N_2) < 1$.

Relation (4) means that $N_2 - N_1 \geqslant O$, i.e., that $(N_2 - N_1)x \geqslant 0$ when $x \geqslant 0$. Similarly, the following result comes from [39]; see also [10,40].

Theorem 1.2. *Let A be a nonsingular matrix with $A^{-1} \geqslant O$ and let $A = M_1 - N_1 = M_2 - N_2$ be two regular splittings. If*

$$M_1^{-1} \geqslant M_2^{-1}, \tag{5}$$

then $\rho(M_1^{-1}N_1) \leqslant \rho(M_2^{-1}N_2) < 1$.

It is not hard to see that, in this context, condition (4) implies (5); see, e.g., [10], and further Lemma 5.9.

When A is singular, several authors have provided examples, where (4) holds, while $\gamma(M_1^{-1}N_1) \not\leqslant \gamma(M_2^{-1}N_2)$; see [5,15]. The following example is from [15].

Example 1.3. Consider the matrix

$$A = \begin{bmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} & 0 \\ -\frac{1}{2} & 1 & 0 & -\frac{1}{2} \\ -\frac{1}{2} & 0 & 1 & -\frac{1}{2} \\ 0 & -\frac{1}{2} & -\frac{1}{2} & 1 \end{bmatrix}$$

and the two regular splittings $A = M_1 - N_1 = M_2 - N_2$ defined by

$$N_1 = \begin{bmatrix} 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad N_2 = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Then $N_1 \leqslant N_2$, but $\gamma(M_1^{-1}N_1) = 1/9 > \gamma(M_2^{-1}N_2) = 0$.

Thus, it has been clear since the early 1980s that a comparison of the convergence factors was hard to obtain, and that conditions of the form (4) and (5) do not suffice. It is the purpose of this paper to present computable conditions that allow the comparison of the convergence factors.

The conditions used in our comparison are similar to (4) and (5), but the inequalities are not interpreted in the usual sense. Instead, we use the partial order induced by a cone $\mathscr{K}_1$ different from the cone of nonnegative vectors $\mathbb{R}^n_+$; see Section 2 and the Appendix. Our contribution also consists of providing a characterization of a cone $\mathscr{K}_1$ for the partial order which can, in certain cases, be computed easily, and therefore the conditions for the comparison can be checked computationally. Furthermore, we provide a new interpretation of weak regular splittings with respect to the partial order induced by the cone $\mathscr{K}_1$. With these $\mathscr{K}_1$-weak regular splittings we provide an equivalent condition for $\gamma(M^{-1}N) < 1$ in the same spirit as in the following well known theorem for the nonsingular case, the proof of which can be found, e.g., in [4,28,38]; see Section 4.

Theorem 1.4. *Let A be nonsingular. Let* $A = M - N = M(I - T)$ *be a weak regular splitting. Then* $\rho(T) < 1$ *if and only if* $A^{-1} = (I - T)^{-1}M^{-1} \geqslant 0$.

2. The cone for the comparison

We say that a matrix $T \in \mathbb{R}^{n\times n}$ is convergent if $\lim_{k\to\infty} T^k$ exists. In this paper, we have in mind the case where $\rho(T) = 1$. There are several conditions that can guarantee that a matrix be convergent; see, e.g., [4,21,25,26,35]. The following result, the proof of which can be found in [22], specifies one of these conditions, cf. [4, Lemma 7.6.9]. It is a powerful tool for analysis and it was used, e.g., in [17,19].

Theorem 2.1. *Let* $T \in \mathbb{R}^{n\times n}$. *T is convergent if and only if*

$$T = P + Z, \quad \text{where } P^2 = P,\ PZ = ZP = O \tag{6}$$

and $\rho(Z) < 1$. *Moreover*, *P is a projection onto* $\mathscr{N}(I - T)$.

It follows from Theorem 2.1 that $\lim_{k\to\infty} T^k = P$. For the case that interests us here, i.e., when $A = M - N$ and $T = M^{-1}N$, the matrix P is a projection onto $\mathscr{N}(A)$. As is well known, an expression for this projection is $P = I - (I - T)^{\#}(I - T)$, where the notation $Q^{\#}$ stands for the (unique) group inverse of Q; see, e.g., [6,21]. Thus, $I - P = (I - T)^{\#}(I - T)$. If $\dim \mathscr{N}(A) = 1$, e.g., when A is irreducible, then

$$P = \hat{x}\hat{e}^{\mathrm{T}}, \quad \text{with} \quad \hat{e}^{\mathrm{T}}\hat{x} = 1, \tag{7}$$

where $\hat{x} \in \mathscr{N}(A)$, and $\hat{e}$ some vector in $\mathbb{R}^n$.

Example 2.2. Consider the matrix

$$T = \begin{bmatrix} 1/2 & 1/3 \\ 1/2 & 2/3 \end{bmatrix} \quad \text{with } \sigma(T) = \{1/6, 1\}.$$

Then $T = P + Z$, where

$$P = \begin{bmatrix} 2/5 & 2/5 \\ 3/5 & 3/5 \end{bmatrix} = \hat{x}\hat{e}^{\mathrm{T}}, \quad \text{with } \hat{x} = e = [1, 1]^{\mathrm{T}} \text{ and } \hat{e} = [2/5, 3/5]^{\mathrm{T}},$$

and

$$Z = \begin{bmatrix} 1/10 & -1/15 \\ -1/10 & 1/15 \end{bmatrix}.$$

One can verify that the relations in (6) hold and that $\rho(Z) = \gamma(T) = 1/6$.

Example 2.3. Consider the matrix $A = I - B$ and the two splittings of Example 1.3. Let $T_0 = B$, $T_i = M_i^{-1}N_i$, $i = 1, 2$. Since A is symmetric, we have $\hat{x} = e \in$

$\mathscr{N}(A)$. Let $T_i = P_i + Z_i$ satisfying (6), $i = 0, 1, 2$. We obtain $P_i = \hat{x}\hat{e}_i^{\mathrm{T}}$, $i = 0, 1, 2$, where $\hat{e}_0^{\mathrm{T}} = [1/4, 1/4, 1/4, 1/4]$, $\hat{e}_1^{\mathrm{T}} = [0, 0, 1/2, 1/2]$, and $\hat{e}_2^{\mathrm{T}} = [0, 1/4, 1/4, 1/2]$.

Remark 2.4. It follows from Example 2.3, that the iteration matrices obtained from different splittings of the same matrix A, may have associated with them totally different projections P_i onto the same subspace $\mathscr{N}(A) = \mathscr{R}(P_i)$.

Given a convergent matrix $T = P + Z$ satisfying (6), the cone which will be used for our comparison is the cone generating the range of the projection $I - P = (I - T)^{\#}(I - T)$. In other words, we will use $\mathscr{K}_1$ such that for every element $u \in \mathscr{R}(I - P)$, there are $v, w \in \mathscr{K}_1$ (usually not unique) such that $u = v - w$, i.e., $\mathscr{K}_1 - \mathscr{K}_1 = \mathscr{R}(I - P)$. (We review the definition of a generating cone in the Appendix.)

Let $\mathscr{K}$ be a generating cone of $\mathbb{R}^n$. Note that we can choose $\mathscr{K}_1 = (I - P)\mathscr{K}$, and similarly, for every cone $\mathscr{K}_1$ generating $\mathscr{R}(I - P)$ we can find $\mathscr{K}$, a generating cone of $\mathbb{R}^n$, such that $\mathscr{K}_1 = (I - P)\mathscr{K}$. Note also that we always have $(I - P)\mathscr{K}_1 = \mathscr{K}_1$, and furthermore $I - P$ is the identity operator on $\mathscr{K}_1$ and on $\mathscr{R}(I - P)$.

It follows from Remark 2.4 that we have to be very specific as to which subspace $\mathscr{R}(I - P)$ we choose to define a generating cone for our comparisons. Note also that in the same way that there are many generating cones for $\mathbb{R}^n$, there is a wide choice of generating cones for $\mathscr{R}(I - P)$. In what follows we characterize the subspace $\mathscr{R}(I - P)$, and one general way of defining generating cones for it. We begin with the case of irreducible A, which implies that $\dim \mathscr{N}(A) = 1$.

It follows from (7) that $P^{\mathrm{T}}\hat{e} = \hat{e}$, i.e., $(I - P)^{\mathrm{T}}\hat{e} = 0$. In this unidimensional null space case we can then characterize $\mathscr{R}(I - P)$ as

$$\mathscr{R}(I - P) = \{x \in \mathbb{R}^n : x^{\mathrm{T}}\hat{e} = 0\}. \tag{8}$$

We can then choose

$$\mathscr{K}_1 = \left\{ x \in \mathbb{R}^n : x = \sum_{i=1}^{n-1} \alpha_i v_i,\ \alpha_i \geqslant 0,\ \ i = 1, \ldots, n-1 \right\}, \tag{9}$$

where the $n - 1$ vectors $v_i \in \mathscr{R}(I - P)$ (i.e., $v_i^{\mathrm{T}}\hat{e} = 0$) are linearly independent, cf. (A.1).

There is a general case where the vector $\hat{e}$ can be known a priori.

Theorem 2.5. *Let A be such that* $\dim \mathscr{N}(A) = 1$. *Let $\hat{x} \in \mathscr{N}(A)$, $e \in \mathscr{N}(A^{\mathrm{T}})$ such that $e^{\mathrm{T}}\hat{x} = 1$. Let $A = M - N = M(I - T)$. Let $T = P + Z$ satisfying* (6). *Then*

$$\mathscr{R}(A) = \mathscr{R}(I - P) \tag{10}$$

if and only if

$$M^{\mathrm{T}}e \in \mathscr{N}(A^{\mathrm{T}}). \tag{11}$$

Proof. Let $\hat{x} \in \mathscr{N}(A)$. Assume that (10) holds, but (11) does not. Then $M^{\mathrm{T}}e = \hat{e} \neq \alpha e$, for any $\alpha \in \mathbb{R}$. On the other hand, $T^{\mathrm{T}}\hat{e} = N^{\mathrm{T}}e = M^{\mathrm{T}}e = \hat{e}$, which implies that $P = \hat{x}\hat{e}^{\mathrm{T}}$. Thus,

$$\mathscr{R}(A) = \{x \in \mathbb{R}^n : x^{\mathrm{T}}e = 0\} \neq \{x \in \mathbb{R}^n : x^{\mathrm{T}}\hat{e} = 0\} = \mathscr{R}(I - P),$$

where the first equality follows from the fact that $e \in \mathscr{N}(A^{\mathrm{T}})$. This contradicts (10). Conversely, if (11) holds, then $P = \hat{x}e^{\mathrm{T}}$ and (10) holds. □

A special case where Theorem 2.5 applies is for the natural splitting $A = I - B$. When B is column stochastic, then $B^{\mathrm{T}}e = e$, and if $B = P + Z$ satisfying (6), then (10) holds. An example of this case is the matrix T_0 considered in Example 2.3, where $\hat{e}_1 = (1/4)e$. Furthermore, if there are two different splittings satisfying (11), then their iteration matrices share the associated projection, cf. Remark 2.4. Example 2.6 illustrates this fact.

We point out that our comparison theorems, presented later in Section 5, are only valid when the two iteration matrices have the same associated projection. If both satisfy condition (11), this is one instance when our comparison theorems apply.

Example 2.6. Let $A = M_1 - N_1 = M_2 - N_2$, where

$$A = \begin{bmatrix} 0 & -2/3 & -1/3 \\ 0 & 1 & -1/3 \\ 0 & -1/3 & 2/3 \end{bmatrix}, \qquad M_1 = \begin{bmatrix} 2/3 & -2/15 & -2/15 \\ 0 & 16/15 & -4/15 \\ 0 & -4/15 & 16/15 \end{bmatrix}$$

and $M_2 = I$. We have that $T_1 = M_1^{-1}N_1 = P + Z_1$, and $T_2 = M_2^{-1}N_2 = P + Z_2$, where

$$P = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$Z_1 = \begin{bmatrix} 0 & -1/6 & -7/12 \\ 0 & 1/12 & 1/6 \\ 0 & 1/12 & 5/12 \end{bmatrix} \quad \text{and} \quad Z_2 = \begin{bmatrix} 0 & -1/3 & -2/3 \\ 0 & 0 & 1/3 \\ 0 & 1/3 & 1/3 \end{bmatrix}.$$

Let us highlight the fact that condition (11) is easily computable. Furthermore, when this condition is satisfied, we can compute $\mathscr{R}(I - P)$ even if we do not know P. In this case, we have $\mathscr{R}(I - P) = \{x \in \mathbb{R}^n : x^{\mathrm{T}}e = 0\}$. For example, when $n = 2$, as in Example 2.2, then $\mathscr{R}(I - P) = \mathscr{K}_1 - \mathscr{K}_1$ where

$$\mathscr{K}_1 = \{\alpha w,\ \alpha \geqslant 0,\ w^{\mathrm{T}} = [1, -1]\}. \tag{12}$$

In the case (2), since $A^{\mathrm{T}}e = 0$, we can choose the generating cone as in (9), where the vectors v_i form a basis of $\mathscr{R}(A)$, e.g., $n - 1$ independent columns of A, cf. (A.1). As we show later in Section 7, this may not be a good choice of the cone $\mathscr{K}_1$, but it is a possible choice.

We discuss now how to produce $\mathscr{K}_1$ when the matrix B in (2) is reducible. Let B be any $n \times n$ column stochastic matrix. It is well known [14, p. 348] that there exists a permutation matrix H such that

$$HBH^{\mathrm{T}} = \begin{bmatrix} G_0 & O & \cdots & O \\ G_1 & F_1 & \cdots & O \\ \cdot & & \cdot & \\ \cdot & & & \cdot \\ \cdot & & & & \cdot \\ G_p & O & \cdots & F_p \end{bmatrix}, \quad p \geqslant 1, \tag{13}$$

where G_0 is an $n_0 \times n_0$ zero-convergent nonnegative matrix, i.e., $\rho(G_0) < 1$, and each of the matrices $F_1, \ldots, F_p$ is an irreducible column stochastic matrix of order $n_j,\ j = 1, \ldots, p$, and $\sum_{j=0}^{p} n_j = n$. Note that the first column need not appear in (13), i.e., the permuted matrix could be block diagonal.

There is a very efficient algorithm developed by Tarjan and with an implementation described in [11] to find the permutation H and the matrices in (13). Our experience shows that its complexity is close to linear. Thus, using Tarjan's algorithm we are able to investigate general stochastic matrices because we are able to check whether a vector belongs to $\mathscr{R}(I - P)$, the subspace "orthogonal" to the Perron eigenspace, or not. Here P is the sum of the Perron projections

$$H^{\mathrm{T}} \begin{bmatrix} O & & & & \\ & \cdot & & & \\ & & P_j & & \\ & & & \cdot & \\ & & & & \cdot \\ & & & & & O \end{bmatrix} H, \tag{14}$$

where P_j is the Perron eigenprojection of the matrix F_j. Consequently, we can construct cone $\mathscr{K}_1$ by setting

$$\mathscr{K}_1 = \{0\} \oplus \mathscr{K}_1^{(1)} \oplus \cdots \oplus \mathscr{K}_1^{(p)},$$

where $\mathscr{K}_1^{(j)}$ is the cone generating $\mathscr{R}(I - P_j)$, e.g., of the form (9), the construction of which has already been demonstrated above for the case of irreducible B.

3. The partial order

Let $\mathscr{K}_1 = (I - P)\mathscr{K}$, with $\mathscr{K} - \mathscr{K} = \mathbb{R}^n$. By definition, the cone $\mathscr{K}_1$ generates a proper subspace, i.e., not the whole space. Therefore, to define a partial order on $\mathbb{R}^n$, we need to restrict the vectors to $\mathscr{R}(I - P)$. Thus, we say that $x \overset{\mathscr{K}_1}{\succeq} y$, $x,\ y \in \mathbb{R}^n$, if $(I - P)(x - y) \in \mathscr{K}_1$. Similarly, a matrix $T \in \mathbb{R}^{n \times n}$ is said to be $\mathscr{K}_1$-nonnegative, denoted $T \overset{\mathscr{K}_1}{\succeq} O$, if

$$(I - P)T(I - P)x \in \mathscr{K}_1 \quad \text{for all} \;\; x \in \mathscr{K}. \tag{15}$$

The partial order thus defined, depends of course on the projection matrix P and on the choice of the cone $\mathscr{K}_1$, but we drop the reference to $\mathscr{K}_1$ in our notation when it is clear from the context which cone it is, and simply write $x \succeq 0$ for $x \in \mathscr{K}_1$ and $T \succeq O$ for $T\mathscr{K}_1 \subset \mathscr{K}_1$.

There is a subtle distinction here between the partial order defined by $\mathscr{K}_1$ on the whole space, and the one defined in the subspace $\mathscr{R}(I - P)$. Thus, the operator T is $\mathscr{K}_1$-nonnegative in $\mathbb{R}^n$ whenever the reduced operator

$$T_P = (I - P)T(I - P)$$

is $\mathscr{K}_1$-nonnegative in $\mathscr{R}(I - P)$. Observe that one does not need to compute T_P in order to check that $T \succeq O$. First of all, condition (15) is equivalent to saying that $(I - P)Ty \in \mathscr{K}_1$ for all $y \in \mathscr{K}_1$, since we can always write $y \in \mathscr{K}_1$ as $(I - P)x$ for some $x \in \mathscr{K}$, and conversely for every $x \in \mathscr{K}$, $y = (I - P)x \in \mathscr{K}_1$. Now we can show that (15) can be verified by checking the usual condition

$$T\mathscr{K}_1 \subset \mathscr{K}_1. \tag{16}$$

To see this, consider (16) valid, i.e., $Ty \in \mathscr{K}_1$, with $y \in \mathscr{K}_1$. Then, by writing $Ty = (I - P)z$, for some z and using the fact that $I - P$ is a projection, we have $(I - P)Ty = (I - P)^2z = (I - P)z = Ty$, so we are in the previous case.

As a corollary to this discussion we have the following general result, used later in the paper.

Lemma 3.1. *Let P be a projection in $\mathbb{R}^n$, and let $\mathscr{K}_1$ be a generating cone of $\mathscr{R}(I - P)$. Then for any matrix T such that $T\mathscr{K}_1 \subset \mathscr{K}_1$, the following holds*:

$$(I - P)T(I - P)x = (I - P)Tx = T(I - P)x \quad \textit{for any}\; x \in \mathscr{K}_1.$$

Example 3.2. Consider the matrix

$$V = \begin{bmatrix} 5 & -1 \\ -2 & 4 \end{bmatrix}.$$

Let $\mathscr{K}_1$ and w be as in (12). It follows that $Vw = 6w \in \mathscr{K}_1$ so that $V \succeq O$, though $V \not\geq O$.

4. $\mathscr{K}_1$-regular splittings

Theorems 1.1, 1.2, and 1.4 remain valid if we interpret each matrix inequality (including the definition of regular and weak regular splittings) with respect to any generating cone $\mathscr{K}$ of $\mathbb{R}^n$. The proofs of the theorems with the new partial orders are either identical or they can be obtained; see [4,16,30,36–38], and also [31, Lemma 1].

We note that in the case of the partial order of Hermitian positive-definite matrices, the corresponding theorems using P-regular splittings, can be found in [24].

Using this order, comparisons for the convergence factor in the Hermitian semidefinite case were recently obtained [8].

Our situation is a bit different since we use a partial order induced by a cone $\mathscr{K}_1$ which does not generate the whole space, but rather a proper subspace. The definitions of the different splittings (e.g., regular splittings) can be given in the same way, but we need to reformulate the convergence and comparison theorems. The following example shows a regular splitting with respect to the cone $\mathscr{K}_1$, which is not a regular splitting (nor weak regular) in the usual sense.

Example 4.1. Let $A = M_2 - N_2$, where

$$A = \begin{bmatrix} 1/2 & -2/3 \\ -1/2 & 2/3 \end{bmatrix} \quad \text{and} \quad M_2 = \begin{bmatrix} 4/3 & 1/3 \\ 2/3 & 5/3 \end{bmatrix}.$$

We have that

$$M_2^{-1} = \frac{1}{6}\begin{bmatrix} 5 & -1 \\ -2 & 4 \end{bmatrix}.$$

Let $\mathscr{K}_1$ and w be as in (12). It follows that $M_2^{-1}w = w$ (see Example 3.2), and $N_2 w = \frac{1}{6}w$, so the splitting is a $\mathscr{K}_1$-regular splitting. (The iteration matrix for this splitting is the matrix T in Example 2.2.)

We begin our development of the theory with a couple of very general results.

Lemma 4.2. *Let $A = M - N = M(I - T)$ be a $\mathscr{K}_1$-weak splitting of A, i.e., such that $T\mathscr{K}_1 \subset \mathscr{K}_1$. Let $1 \notin \sigma(Z)$, where $T = P + Z$, $PZ = ZP = O$. Then*

$$(M^{-1}A)^{\#} = (I - T)^{\#} = (I - P)(I - Z)^{-1}. \tag{17}$$

Proof. We easily check the validity of the three relations defining the group inverse of $I - T$.

(i) $(I - T)(I - P)(I - Z)^{-1} = (I - P)(I - Z)^{-1}(I - T) = I - P.$

(ii) $$(I - T)^2(I - P)(I - Z)^{-1} = (I - T)(I - P - Z)(I - P)(I - Z)^{-1} = I - T.$$

(iii) $$(I - T)\left[(I - T)^{\#}\right]^2 = (I - P)(I - Z)^{-1}(I - T)(I - P)(I - Z)^{-1} = (I - P)(I - Z)^{-1} = (I - T)^{\#}. \qquad \square$$

Observe that in relation (i), we have that $(I-T)^{\#}(I-T)=(I-T)(I-T)^{\#}=I-P$, which is the identity in the subspace $\mathscr{R}(I-P)$ and in the generating cone $\mathscr{K}_1$.

Theorem 4.3. *Let A be singular. Let $A=M-N=M(I-T)$ be a $\mathscr{K}_1$-weak splitting, with $T=P+Z,\ PZ=ZP=O,\ P^2=P,\ 1\notin\sigma(Z)$, and $\mathscr{K}_1$ a cone generating $\mathscr{R}(I-P)$. Then $\gamma(T)=\rho(Z)<1$ if and only if $(I-T)^{\#}\succeq O$, i.e., $(I-T)^{\#}$ is $\mathscr{K}_1$-nonnegative.*

Proof. Assume first that $\gamma(T)=\rho(Z)<1$. Since, by hypothesis, $Z\mathscr{K}_1=T\mathscr{K}_1\subset\mathscr{K}_1$, using (17), we see that

$$(I-T)^{\#}=(I-P)\sum_{k=0}^{\infty}Z^k\succeq O. \tag{18}$$

For the necessity, using the fact that $Z\mathscr{K}_1\subset\mathscr{K}_1$, there is a Perron eigenvector $x=(I-P)x\in\mathscr{K}_1$ for which $Zx=\rho(Z)x$. Thus, the relation

$$\frac{1}{1-\rho(Z)}x=(I-T)^{\#}x\in\mathscr{K}_1$$

implies that $\rho(Z)=\gamma(T)<1$ and this completes the proof. □

Consider the reduced operator $A_P=(I-P)A(I-P)$, i.e., the restriction of the matrix A acting on the (proper) subspace $\mathscr{R}(I-P)$. We first observe that in spite of A being singular, this reduced operator is nonsingular on $\mathscr{R}(I-P)$. To see this, it suffices to see that a linear system associated with this matrix has a unique solution in $\mathscr{R}(I-P)$. This follows from the fact that if $w\in\mathscr{N}(A)$, then $Pw=w$ and thus $(I-P)w=0$, so the null space of A is reduced to the zero element in $\mathscr{R}(I-P)$.

Consider now the matrix

$$A^{-}=(I-T)^{\#}M^{-1}(I-P).$$

It is an $\{1\}$-inverse of A [4,6]. In fact, the following two relations hold, and can be verified directly.

$$AA^{-}=I-P,\qquad A^{-}AA^{-}=A^{-}. \tag{19}$$

Remark 4.4. It follows from (19) that if $b\in\mathscr{R}(I-P)$, $A^{-}b$ is the only element in $\mathscr{R}(I-P)$ that solves $Ax=b$.

The following theorem is the reformulation of Theorem 1.4. Several authors have attempted to present counterparts to Theorem 1.4 for singular systems, using alternative definitions of regular splittings; see [21,25,29]. In these references, the conditions defining the regular splittings are not easily verifiable. We believe that this is the first time that an analogous result is produced in which this hypothesis can be checked; see, e.g., Example 4.1.

Theorem 4.5. *Let A be singular. Let* $A = M - N = M(I - T)$ *be a* $\mathscr{K}_1$*-weak regular splitting. Let* $T = P + Z$ *with* $PZ = ZP = O$, $P^2 = P$, $1 \notin \sigma(Z)$, *and* $\mathscr{K}_1$ *a cone generating* $\mathscr{R}(I - P)$. *Then* $\gamma(T) < 1$ *if and only if* $A^- = (I - T)^{\#}M^{-1}(I - P) \succeq O$, *i.e.*, A^- *is* $\mathscr{K}_1$*-nonnegative.*

Proof. Assume first that $\gamma(T) = \rho(Z) < 1$. The sufficiency follows by multiplying (18) on the right by $M^{-1}(I - P) \succeq O$. On the other hand, assume that $A^- \succeq 0$. Since $Z \succeq 0$ and $M^{-1} \succeq O$, and for every *m*,

$$(I + Z + Z^2 + \cdots + Z^m)(I - Z) = I - Z^{m+1}, \tag{20}$$

we have that

$$\begin{aligned}0 &\preceq (I + Z + Z^2 + \cdots + Z^m)(I - P)M^{-1}(I - P)\\ &= (I + Z + Z^2 + \cdots + Z^m)(I - Z)(I - Z)^{-1}(I - P)M^{-1}(I - P)\\ &= (I - Z^{m+1})A^- \preceq A^-,\end{aligned}$$

where the last inequality follows from the assumption $A^- \succeq 0$. The theorem follows now as in the nonsingular case, by noting that the partial sum (20) is bounded in the partial order induced by $\mathscr{K}_1$, since $Z \succeq 0$, the partial sum is convergent, thus $\gamma(T) = \rho(Z) < 1$. □

5. Comparison theorems

We follow the spirit and ideas used for the nonsingular case in our paper [18]. We begin with a new simple lemma.

Lemma 5.1. *Let* $A = M - N$ *be a convergent* $\mathscr{K}_1$*-weak splitting, and let* $T = M^{-1}N$. *Then*

$$\rho((I - T)^{\#}T) = \frac{\gamma(T)}{1 - \gamma(T)}\,. \tag{21}$$

Proof. Let μ and *x* be an eigenpair of *T*, i.e., $Tx = \mu x,\ x \neq 0$. First consider the Perron eigenvector $\hat{x}$, i.e., $\mu = 1$. By (17),

$$(I - T)^{\#} = (I - P)(I - T)^{\#}(I - P)$$

and since $(I - P)\hat{x} = 0$, we have that $(I - T)^{\#}T\hat{x} = 0$. For the other eigenvectors, i.e., for $\mu \neq 1$, it follows that

$$(I - T)^{\#}Tx = \frac{\mu}{1 - \mu}x.$$

Therefore, since $T\mathscr{K}_1 \subset \mathscr{K}_1$,

$$\rho((I - T)^{\#}T) = \max\left\{\frac{\mu}{1 - \mu} : \mu \in \sigma(T),\ \ 1 \neq \mu > 0\right\}.$$

Since the function

$$f(\mu)=\frac{\mu}{1-\mu}, \tag{22}$$

is increasing for $\mu < 1$, the lemma follows. □

Lemma 5.2 [18, Corollary 3.2]. *Let $V \succeq O$, and let $x \succeq 0$ be such that $Vx-\alpha x \succeq 0$. Then $\alpha \leqslant \rho(V)$.*

Theorem 5.3. *Let A be singular. Let $A=M_1-N_1=M_1(I-T_1)=M_2-N_2=M_2(I-T_2)$ be two (convergent) $\mathscr{K}_1$-regular splittings, where $\mathscr{K}_1$ is the cone generating $\mathscr{R}(I-P)$ and $T_1=P+Z_1$, $P^2=P$, $PZ_1=Z_1P=O$, $\rho(Z_1)<1$, $T_2=P+Z_2$, $P^2=P$, $PZ_2=Z_2P=O$, $\rho(Z_2)<1$. If*

$$N_2 \succeq N_1, \tag{23}$$

then

$$\gamma(T_1) \leqslant \gamma(T_2). \tag{24}$$

Proof. By Theorem 4.5, $A_1^- = (I-T_1)^{\#}M_1^{-1}(I-P) \succeq O$, and $A_2^- = (I-T_2)^{\#} M_2^{-1}(I-P) \succeq O$. Let $x \succeq 0$ be the Perron eigenvector of Z_1, i.e.,

$$T_1x=\gamma(T_1)x=Z_1x=\rho(Z_1)x, \quad x\in\mathscr{K}_1, \quad x\neq 0.$$

By hypothesis, $N_1x \succeq 0$, and $N_2x \succeq 0$. By Remark 4.4, we have that $A_2^-N_1x = A_1^-N_1x$. From (23) it follows that

$$\begin{aligned}0 &\preceq A_2^-(N_2x-N_1x)=A_2^-N_2x-A_1^-N_1x\\ &=(I-T_2)^{\#}M_2^{-1}(I-P)N_2x-(I-T_1)^{\#}M_1^{-1}(I-P)N_1x\\ &=(I-T_2)^{\#}T_2x-\frac{\gamma(T_1)}{1-\gamma(T_1)}x,\end{aligned} \tag{25}$$

where the last equality follows from Lemma 3.1 and from (17). The theorem now follows from Lemmas 5.2 and 5.1, and the fact that the function (22) is increasing. □

We note that the parts of the proof where the projections associated with T_1 and T_2 must be the same are in the $\mathscr{K}_1$-nonnegativity of A_1^- and A_2^-, and in the use of Lemma 3.1 in (25).

Remark 5.4. Theorem 5.3 is valid with slightly weaker hypotheses, using the same proof. Namely, that the splittings are convergent $\mathscr{K}_1$-weak splittings and that if $x \succeq 0$ is the Perron eigenvector of Z_1, then

$$N_1x \succeq 0 \ \text{ and } \ N_2x \succeq 0, \tag{26}$$

cf. [18, Theorem 3.5]. A similar proof can be obtained if (26) holds for x being the Perron eigenvector of Z_2 and lying in the interior of $\mathscr{K}_1$. Furthermore, one can extend the result to strict inequality in (24) with suitable hypotheses on the splittings; see [18,40].

Example 5.5. Let $A = M_1 - N_1 = M_2 - N_2$, $T_1 = M_1^{-1}N_1$, and $T_2 = M_2^{-1}N_2$, where A and M_2 are as in Example 4.1, and

$$M_1 = \begin{bmatrix} 5/4 & 1/3 \\ 2/3 & 19/12 \end{bmatrix}.$$

Note that both $M_1^{\mathrm{T}}e$ and $M_2^{\mathrm{T}}e$ are multiples of $e \in \mathscr{N}(A^{\mathrm{T}})$. Let $\mathscr{K}_1$ and w be as in (12). It follows that $M_1^{-1}w = \frac{12}{11}w \in \mathscr{K}_1$ and $N_1 w = \frac{1}{12}w \in \mathscr{K}_1$, so the two splittings are $\mathscr{K}_1$-regular splittings. Furthermore, $(N_2 - N_1)w = \frac{1}{12}w$, and thus $N_2 \succeq N_1$, which implies $\gamma(T_1) \leqslant \gamma(T_2)$. Indeed, $\gamma(T_1) = 1/11 < \gamma(T_2) = 1/6$.

Theorem 5.6. *Let A be singular. Let $A = M_1 - N_1 = M_1(I - T_1) = M_2 - N_2 = M_2(I - T_2)$ be two (convergent) $\mathscr{K}_1$-regular splittings, where $\mathscr{K}_1$ is the cone generating $\mathscr{R}(I - P)$ and $T_1 = P + Z_1$, $P^2 = P$, $PZ_1 = Z_1P = O$, $\rho(Z_1) < 1$, $T_2 = P + Z_2$, $P^2 = P$, $PZ_2 = Z_2P = O$, $\rho(Z_2) < 1$. If*

$$M_1^{-1} \succeq M_2^{-1}, \tag{27}$$

then $\gamma(T_1) \leqslant \gamma(T_2)$.

Proof. If $\gamma(T_1) = 0$, there is nothing to prove. Since $\mathscr{K}_1$ is the cone generating $\mathscr{R}(I - P)$, and by hypothesis $Z_1\mathscr{K}_1 = T_1\mathscr{K}_1 \subset \mathscr{K}_1$, there is a Perron eigenvector $x = (I - P)x \in \mathscr{K}_1$ for which $T_1x = Z_1x = \rho(Z_1)x = \gamma(T_1)x \succeq 0$. Then,

$$M_1x = \frac{1}{\gamma(T_1)}N_1x \succeq 0 \tag{28}$$

and

$$Ax = M_1(I - T_1)x = \frac{1 - \gamma(T_1)}{\gamma(T_1)}N_1x \succeq 0.$$

Using (27), it follows that

$$0 \preceq (M_1^{-1} - M_2^{-1})Ax = (I - T_1)x - (I - T_2)x = T_2x - \gamma(T_1)x. \tag{29}$$

Since

$$Px = 0, \tag{30}$$

from (29), we have that $Z_2x = T_2x \succeq \gamma(T_1)x$. Thus, applying Lemma 5.2 to Z_2, we have that $\gamma(T_2) = \rho(Z_2) \geqslant \gamma(T_1)$. $\square$

We note that the hypothesis of the common projection is used in (30).

Remark 5.7. Theorem 5.6 is valid with weaker hypotheses, using the same proof. Namely, that the splittings are convergent $\mathscr{K}_1$-weak splittings and that if $x \succeq 0$ is the Perron eigenvector of Z_1, then $N_1 x \succeq 0$; see (28). A similar proof can be used in which the hypotheses are convergent $\mathscr{K}_1$-weak splittings, and if $z \succeq 0$ is the Perron eigenvector of Z_2, with z in the interior of $\mathscr{K}_1$, then $N_2 z \succeq 0$; cf. [18, Theorem 3.11]. Here also, strict inequalities can be obtained essentially with the same proof, with suitable hypotheses.

Example 5.8. Let $A = M_1 - N_1 = M_2 - N_2$, $T_1 = M_1^{-1} N_1$ and $T_2 = M_2^{-1} N_2$, be as in Example 2.6. Let

$$\mathscr{K}_1 = \{\alpha v_1 + \beta v_2,\ \alpha,\ \beta \geqslant 0,\ v_1 = [1, 0, -1]^{\mathrm{T}},\ v_2 = [1, -1, 0]^{\mathrm{T}}\}, \tag{31}$$

cf. (A.1). One can check that the two splittings are $\mathscr{K}_1$-regular splittings, e.g., $N_2 v_1 = \frac{1}{3} v_1 + \frac{1}{3} v_2$, $N_2 v_2 = \frac{1}{3} v_1$. Furthermore, $M_1^{-1} \succeq M_2^{-1}$, and $\gamma(T_1) = (3 + \sqrt{6})/2 = 0.454 < \gamma(T_2) = (1 + \sqrt{5})/6 = 0.539$.

We conclude the section showing, in a way very similar to the nonsingular case, that condition (23) implies condition (27). We remark that the usual hypothesis of $A^{-1} \geqslant O$ is not needed; in fact A need not be nonsingular.

Lemma 5.9. *Let $A = M_1 - N_1 = M_2 - N_2$ be two $\mathscr{K}_1$-weak regular splittings. If $N_2 \succeq N_1$, then $M_1^{-1} \succeq M_2^{-1}$.*

Proof. We have

$$O \preceq N_2 - N_1 = M_2 - M_1 = M_2\,(M_1^{-1} - M_2^{-1})\,M_1.$$

Multiply the last equation by $M_2^{-1} \succeq O$ on the left, and by $M_1^{-1} \succeq O$ on the right and obtain the lemma. □

6. Nearly completely decomposable stochastic matrices

Consider system (1), in the case (2), with $b = 0$. We are going to show that suitable splittings can be found utilizing knowledge of the magnitude of some elements of B. We demonstrate this fact on the example of *nearly completely decomposable stochastic matrices*; see, e.g., [33]. Let B be an $n \times n$ nearly completely decomposable irreducible column stochastic matrix, i.e., let B satisfy (2) and be given in its block form

$$B = \begin{bmatrix} B_{11} & . & . & . & B_{1p} \\ & & & & \\ . & . & . & . & . \\ & & & & \\ B_{p1} & . & . & . & B_{pp} \end{bmatrix} \tag{32}$$

with square diagonal blocks, where the magnitude of the elements in blocks B_{jk}, $j \neq k$, do not exceed the value $\varepsilon > 0$ which is assumed to be small.

A class of widely used methods to compute the stationary probability vectors of such matrices are some aggregation/disaggregation iterations; see, e.g., [2,7,20,27,32–34]. Various aggregation/disaggregation algorithms reduce the original problem of finding a stationary probability vector to a similar problem on the coarse level, i.e., of determining a stationary probability vector of the aggregated matrix $\tilde{B}$. One of the possible ways to aggregate involves reducing each of the blocks in (32) to a single scalar element of the aggregated matrix. In such a case, matrix $\tilde{B}$ is elementwise nearly completely decomposable, i.e. the following relations hold:

$$\tilde{B} = I - \tilde{B}_1 + \tilde{B}_2$$

with

$$\tilde{B}_1 = \operatorname{diag}\{\tilde{b}^1_{11}, \ldots, \tilde{b}^1_{pp}\}, \quad 0 < \tilde{b}^1_{jj} \leqslant \kappa\varepsilon, \quad j = 1, \ldots, p$$

and

$$\tilde{B}_2 = (\tilde{b}^2_{jk}), \quad 0 \leqslant \tilde{b}^2_{jk} \leqslant \tau\varepsilon, \quad j, k = 1, \ldots, p,$$

where $\kappa,\ \tau \in \mathbb{R}$ are independent of ε.

Let us write $\tilde{A} = I - \tilde{B}$. One needs to compute the positive p-vector $\tilde{x} \in \mathscr{N}(\tilde{A})$ normalized so that $\tilde{x}^{\mathrm{T}}e = 1, e^{\mathrm{T}} = [1, \ldots, 1] \in \mathbb{R}^p$. We may also iterate on the coarse level for this purpose. Let us choose the following splittings:

$$\tilde{A} = \delta I - \tilde{G} = \tilde{M}_j - \tilde{N}_j, \quad j = 1, 2,$$

where

$$\begin{aligned} \tilde{M}_1 &= (\alpha + \delta)I, \quad \tilde{N}_1 = \alpha I + \tilde{G}, \\ \tilde{M}_2 &= (\alpha + \beta + \delta)I, \quad \tilde{N}_2 = (\alpha + \beta + \delta)I + \tilde{G} \end{aligned}$$

with $\alpha > \delta$.

Let $\mathscr{K}_1 = (I - P)\mathscr{K}_1 \subset \mathbb{R}^p$, where $\tilde{B} = P + Z,\ P^2 = P,\ PZ = ZP = O$, $1 \notin \sigma(Z)$. It is obvious that the above splittings are $\mathscr{K}_1$-regular if $\alpha > 0$ is sufficiently large and $\beta \geqslant 0$. Let $\tilde{M}_1^{-1}\tilde{N}_1 = T_1 = T(\alpha),\ \tilde{M}_2^{-1}\tilde{N}_2 = T_2 = T(\alpha + \beta)$ and since the peripheral spectrum

$$\sigma_\pi(T(\alpha)(I - P)) = \{\lambda \in \sigma(T(\alpha)), |\lambda| = \gamma(T(\alpha)\},$$

consists just of a single point $\gamma(T(\alpha))$, the matrix $T(\alpha)$ is convergent for $\alpha > 0$ large enough. Furthermore, $\tilde{N}_2 - \tilde{N}_1 = \beta I$, i.e., $\{\tilde{N}_2 - \tilde{N}_2\}\mathscr{K}_1 \subset \mathscr{K}_1$. It follows then that for $\alpha > 0$ sufficiently large, by Theorem 5.3:

$$\gamma(T(\alpha)) = \gamma(T_1) \leqslant \gamma(T_2) = \gamma(T(\alpha + \beta)), \quad \beta \geqslant 0. \tag{33}$$

Relation (33) can be strengthened as mentioned in Remark 5.4.

It follows as a consequence of (33) that for splittings of this form, we have certain monotonicity properties, and we can choose the parameters in such a way that the convergence factor is minimized. As an elementary illustration let us consider for $\kappa > 0$, $\rho > 0$,

$$\tilde{B} = \begin{bmatrix} 1-\rho\varepsilon & \kappa\varepsilon \\ \rho\varepsilon & 1-\kappa\varepsilon \end{bmatrix}, \quad \text{and thus} \quad \tilde{A} = I - \tilde{B} = \varepsilon \begin{bmatrix} \rho & -\kappa \\ -\rho & \kappa \end{bmatrix}.$$

Setting

$$\tilde{G} = \varepsilon \begin{bmatrix} \kappa & \kappa \\ \rho & \rho \end{bmatrix},$$

we can define the splittings

$$M(\alpha) = (\alpha + \delta)I, \quad N(\alpha) = \alpha I + \tilde{G}, \quad \delta = \varepsilon(\kappa + \rho)$$

and the iteration matrices $T(\alpha) = M(\alpha)^{-1}N(\alpha)$, $\alpha \geqslant \delta$. We see that $\gamma(T(\alpha)) < 1$ if $\alpha \geqslant 0$ and that $\min\{\gamma(T(\alpha)) : \alpha \in \mathbb{R}\} = \gamma(T(0)) = 0$.

Let us further specify κ and ρ, by setting $\kappa = \rho = 1$, then

$$\tilde{A} = (\alpha + \varepsilon)I - \alpha I - \varepsilon \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = (\alpha + \varepsilon)I - (\alpha I + \varepsilon\tilde{G}), \quad \alpha \geqslant 0,$$

is a $\mathscr{K}_1$-regular splitting of $\tilde{A}$ for $\alpha \geqslant \varepsilon$, but since $\tilde{G}\mathscr{K}_1 = -\mathscr{K}_1$, $\mathscr{K}_1$ as in (12), this splitting is not $\mathscr{K}_1$-weak for $0 < \alpha < \varepsilon$. It follows that:

$$\gamma(T(\alpha)) \leqslant \gamma(T(\alpha + \beta)), \quad \varepsilon \leqslant \alpha \leqslant \alpha + \beta, \quad \beta \geqslant 0,$$

and we see that $\gamma(T(\varepsilon)) = 0 = \min\{\gamma(T(\alpha)) : \alpha \in \mathbb{R}\}$.

7. Different comparisons

In Example 4.1, we showed a $\mathscr{K}_1$-regular splitting which is not a regular splitting with respect to $\mathbb{R}^2_+$. Similarly, a $\mathscr{K}_1$-regular splitting may not be regular with respect to another cone $\mathscr{K}_2$ which generates the *same* subspace $\mathscr{R}(I - P)$, as shown in the following example.

Example 7.1. Let

$$\mathscr{K}_2 = \{\alpha w_1 + \beta w_2,\ \alpha,\ \beta \geqslant 0,\ w_1 = [-1, 0, 1]^{\mathrm{T}},\ w_2 = [1, -1, 0]^{\mathrm{T}}\}.$$

It can be seen that $w_1 = -v_1$ and $w_2 = v_2$ in (31), and thus this cone generates the same subspace as $\mathscr{K}_1$. Consider A as in Example 5.8 and the second splitting there. The matrix $T = I - A \overset{\mathscr{K}_1}{\geq} O$, but $Tw_1 = [-2/3, 1/3, 1/3]^{\mathrm{T}} \notin \mathscr{K}_2$, and furthermore $Tw_2 = [1/3, 0, -1/3]^{\mathrm{T}} \notin \mathscr{K}_2$. Thus, $A = I - T$ is not a $\mathscr{K}_2$-weak splitting.

Examples 5.8 and 7.1 illustrate the fact that given a splitting $A = M - N$, one may or may not be able to find the appropriate cone $\mathscr{K}_1$ for which this splitting is $\mathscr{K}_1$-weak regular, and thus to apply Theorem 4.5, or the comparison theorems in

Section 5. However, if one does not find the appropriate cone, this does not mean that two splittings cannot be compared using another cone.

Example 7.2. Consider the 4×4 matrix A from Example 1.3 and the following two splittings of it, $A = M_1 - N_1 = M_2 - N_2$:

$$N_1 = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}, \qquad N_2 = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} & 0 \\ \frac{1}{4} & \frac{1}{2} & 0 & \frac{1}{4} \\ \frac{1}{4} & 0 & \frac{1}{2} & \frac{1}{4} \\ 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{bmatrix}.$$

It follows immediately that $D = N_1 - N_2$ is neither nonnegative nor nonpositive in the usual sense, i.e., they cannot be compared using the usual nonnegative cone $\mathbb{R}^n_+$. They do compare though with respect to the cone (9) with $v_1^{\mathrm{T}} = [1, 0, 0, -1]$, $v_2^{\mathrm{T}} = [0, 1, -1, 0]$, and $v_3^{\mathrm{T}} = [1, -1/2, -1/2, 0]$. Indeed we have $Dv_1 = \frac{1}{2}v_2$, $Dv_2 = \frac{1}{2}v_1$, and $Dv_3 = \frac{1}{4}v_2$, and thus $N_1 \succeq N_2$. It can be checked that these two splittings are $\mathscr{K}_1$-weak regular (but not regular!), and thus, by Theorem 5.3 (with the hypothesis as in Remark 5.4), $\gamma(T_2) \leqslant \gamma(T_1)$. The computation of the convergence factors gives $\gamma(T_2) = 1/3$ and $\gamma(T_1) = 1/2$.

We point out that the difference matrix D of Example 7.2 is symmetric but not positive (semi)definite. Therefore, the two splittings cannot be compared using the Hermitian positive definite partial order, i.e., the splittings do not satisfy the hypotheses of the comparison theorems in [8].

Example 7.2 illustrates that, as in the nonsingular case, not every pair of splittings is comparable in the sense of satisfying either (4) or (5). In the singular case, we can also have pairs of splittings $A = M_1 - N_1 = M_2 - N_2$, such that neither (23) nor (27) hold. The situations in the examples presented in [5,15] are not of this kind though. Our theorems do not apply to the example in [5] since the Gauss–Seidel splitting in that example is not convergent, i.e., the convergence factor γ is equal to one. In the case of Example 1.3, which comes from [15], the splittings have associated projection matrices which are not equal, as shown in Example 2.3, and thus, the hypotheses of our comparison theorems are not fulfilled.

When A is nonsingular, there is a unique splitting $A = M - N$ corresponding to the iteration matrix $T = M^{-1}N$. In the singular case, there are many splittings corresponding to the same iteration matrix [3, Theorem 2.2]. In the case that $A^{\mathrm{T}}e = 0$, and $\dim \mathscr{N}(A^{\mathrm{T}}) = 1$, the splittings $A = \tilde{M} - \tilde{N} = M - N$ have the same iteration matrix $M^{-1}N = \tilde{M}^{-1}\tilde{N}$ if $\tilde{M}^{-1} = M^{-1} + ve^{\mathrm{T}}$, for some vector v. So one could imagine that if two splittings do not satisfy (27), we can choose another splitting with the same iteration matrix to see if the new pair is comparable. Unfortunately, this is not possible, since for any $w \in \mathscr{K}_1$, $e^{\mathrm{T}}w = 0$, and thus $\tilde{M}^{-1}w = M^{-1}w$.

We end the section by observing that the methods of comparison presented for consistent singular systems can be applied to inconsistent systems as well. In such a case, the iteration (3) yields

$$\begin{aligned} x_{(k+1)} &= Px_{(0)} + Z^{k+1}x_{(0)} + \sum_{t=0}^{k} T^t c \\ &= Px_{(0)} + Z^{k+1}x_{(0)} + (k+1)Pc + (I - Z^{k+1})(I - Z)^{-1}(I - P)c. \end{aligned}$$

It follows that

$$\hat{x} - T\hat{x} = (I - T)\left[Px_{(0)} + (I - Z)^{-1}(I - P)c\right] = (I - P)c$$

and

$$\lim_{k\to\infty} \|y_{(k+1)} - \hat{x}\| = 0,$$

where

$$y_{(k+1)} = x_{(k+1)} - k\left(x_{(k+1)} - x_{(k)}\right), \qquad \hat{x} = Px_{(0)} + (I - Z)^{-1}(I - P)c;$$

see [12]. Furthermore,

$$\|Pc\|_{\mathrm{P}} = \|M^{-1}\left(A\hat{x} - b\right)\|_{\mathrm{P}} = \min\left\{\|M^{-1}\left(Ax - b\right)\|_{P} : x \in \mathbb{R}^n\right\}.$$

where $\|\cdot\|_{\mathrm{P}}$ denotes the norm on $\mathbb{R}^n$ defined by

$$\|x\|_{\mathrm{P}} = \|Px\| + \|(I - P)x\|,$$

where $\|\cdot\|$ is any norm on $\mathbb{R}^n$.

8. Concluding remarks

We have solved the longstanding problem of establishing theorems comparing the convergence factors of iterative methods for (generally nonsymmetric) singular matrices.

The new theory developed uses a partial order which depends on the projection associated with the iteration matrix. Thus, when comparing two splittings, i.e., two iterative methods, the two iteration matrices need to have the same projection. In many cases, this is the case, and in particular when they satisfy condition (11), this can be easily checked. This condition is a natural one in the case of splittings for Markov chains. The projection matrix need not be known in order to check this condition. Neither is it needed to compute the cone defining the partial order.

With this new theory a new situation arises, a kind of inverse problem. If a splitting is not regular with respect to a certain partial order, one can try to look for the appropriate cone for which it becomes regular. Similarly, if two splittings cannot be compared using a certain partial order, a new cone can be sought so that they are comparable.

Appendix A. The generating cone

Let $\mathscr{E}$ be a real Banach space, $\mathscr{E}'$ its dual and $\mathscr{B}(\mathscr{E})$ the space of all bounded linear operators mapping $\mathscr{E}$ into itself. We do not distinguish between the norms of these spaces writing simply $|\cdot|$.

A normal cone $\mathscr{K}$ is a subset of $\mathscr{E}$ with the following properties:

(i) $\mathscr{K} + \mathscr{K} \subset \mathscr{K}$,
(ii) $\alpha\mathscr{K} \subset \mathscr{K}$ for $\alpha \geqslant 0$,
(iii) $\mathscr{K} \cap (-\mathscr{K}) = \{0\}$,
(iv) $\bar{\mathscr{K}} = \mathscr{K}$, where $\bar{\mathscr{K}}$ denotes the norm-closure of $\mathscr{K}$, and
(v) $\exists\, \sigma > 0$ such that for $x,\ y \in \mathscr{K}$ one has $\|x + y\| \geqslant \sigma\|x\|$.

We say that $\mathscr{K}$ is generating if $\mathscr{E} = \mathscr{K} - \mathscr{K}$. The typical example is $\mathscr{E} = \mathbb{R}^n$ (in which case property (v) is automatically satisfied), and a generating cone is

$$\begin{aligned}\mathscr{K} = \mathbb{R}^n_+ &= \{x \in \mathbb{R}^n : x \geqslant 0\}\\ &= \left\{x \in \mathbb{R}^n : x = \sum_{i=1}^{n} \alpha_i e_i,\ \alpha_i \geqslant 0, i = 1, \ldots, n\right\},\end{aligned} \tag{A.1}$$

where e_i is the standard *i*th canonical vector, i.e., the *i*th column of the identity.

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