Topics in Monge-Ampère type Equations I

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II General nonlinear elliptic pdes

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I

Some important elliptic pdes and where they appear

A. POTENTIAL THEORY

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  this means that the force felt at the point x by the presence of Ω is $F(x)$.

- $F(x)$ is conservative $\Longrightarrow$ $F(x) = D\phi(x)$ where $\phi(x)$ is the Newtonian potential

$$\phi(x) = G \int_\Omega \rho(y) \frac{1}{|y - x|^{n-2}} \, dy.$$

- $\phi \in C^1(\mathbb{R}^n)$, $\phi \in C^2(\mathbb{R}^n \setminus \bar{\Omega})$

$$\Delta\phi(x) = 0, \qquad \text{for } x \in \mathbb{R}^n \setminus \bar{\Omega} \text{ (Laplace's equation).}$$

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- If in addition, ρ is locally Hölder continuous in Ω, then $\phi \in C^2(\Omega)$ and

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- These are perhaps the two most important elliptic equations, they serve as a guide for many problems and appear in many contexts.

Suppose $u(x)$ is a smooth function. By the Taylor's expansion

$$u(x) = u(y) + Du(y) \cdot (x-y) + \frac{1}{2}\langle D^2u(y)\,(x-y), x-y\rangle + h(x,y)$$

where $\dfrac{|h(x,y)|}{|x-y|^2} \to 0$ as $x \to y$.

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where $\dfrac{|h(x,y)|}{|x-y|^2} \to 0$ as $x \to y$.Integrating this identity over the ball $B_r(y)$ yields

$$\int_{B_r(y)} u(x)\,dx = |B_r(y)|\,u(y) + \sum_{i=1}^{n} \int_{B_r(y)} u_i(y)\,(x_i - y_i)\,dx$$

$$+ \frac{1}{2}\sum_{i,j=1}^{n} \int_{B_r(y)} u_{ij}(y)\,(x_i - y_i)(x_j - y_j)\,dx + \int_{B_r(y)} h(x, y)\,dx.$$

By symmetry, the second summand on the RHS is zero and the third summand equals

$$\frac{1}{2}\Delta u(y) \int_{B_r(y)} |x-y|^2 \, dx = C_n \, |B_r(y)| \, \Delta u(y) \, r^2,$$

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Showing the connection between harmonicity and the mean value property. This formula contains the strong maximum principle: u harmonic in Ω connected, and $\exists\, x_0 \in \Omega$ such that $\max_{\bar{\Omega}} u = u(x_0)$ $\Longrightarrow$ u is constant in Ω. In particular, if u is continuous in $\bar{\Omega}$, then u attains its maximum on the bdry.

An important and basic problem in pdes is to solve the Dirichlet problem

$$\Delta u = 0, \qquad \text{in } \Omega$$

$$u = \phi, \qquad \text{on } \partial\Omega$$

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- Consider the functional

$$I[w] = \int_\Omega L(Dw(x), w(x), x)\, dx$$

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- Problem: minimize $I[w]$ among all w's such that $w = g$ on $\partial\Omega$.

- The minimizer u to this problem satisfies a nonlinear pde.

Since u is a minimizer $\Longrightarrow$ the minimum of

$$i(t) = I[u + tv]$$

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$$\int_\Omega \left[\sum_{i=1}^n L_{p_i}(Du, u, x)\, v_{x_i} + L_z(Du, u, x)\, v \right] dx = 0$$

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since v has compact support in Ω, it follows integrating by parts that

$$\int_\Omega \left[-\sum_{i=1}^{n} \Big(L_{p_i}(Du, u, x) \Big)_{x_i} + L_z(Du, u, x) \right] v\, dx = 0$$

for all such v.

Therefore the minimizer u satisfies Euler-Lagrange pde (in general nonlinear):

$$-\sum_{i=1}^{n}\left(L_{p_i}(Du,u,x)\right)_{x_i}+L_z(Du,u,x)=0, \qquad \text{in } \Omega.$$

Exercise 2: see proof in Evans' book

Important examples arise taking various L's

1. **LAPLACE EQUATION.**

If $L(p, z, x) = |p|^2$, then the functional I is

$$I[w] = \int_\Omega |Dw|^2 \, dx,$$

and the Euler-Lagrange pde satisfied by the minimizer is Laplace's equation

$$\Delta u = 0, \text{in } \Omega.$$

2. **DIVERGENCE ELLIPTIC EQS.**

If $L(p, z, x) = \sum_{i,j=1}^{n} a_{ij}(x)\, p_i\, p_j - z\, f(x)$, then the functional I is

$$I[w] = \int_\Omega \left(\sum_{i,j=1}^{n} a_{ij}(x)\, w_{x_i}\, w_{x_j} - w\, f(x) \right) dx,$$

and the Euler-Lagrange pde satisfied by the minimizer is the divergence form pde

$$-\sum_{i,j=1}^{n} \left(a_{ij}\, u_{x_j} \right)_{x_i} = f, \qquad \text{in } \Omega.$$

3. **MINIMAL SURFACE EQ.**

If $L(p,z,x) = (1+|p|^2)^{1/2}$, then the functional I is

$$I[w] = \int_\Omega (1+|Dw|^2)^{1/2}\,dx,$$

which represents the area of the graph of the function w over Ω. The Euler-Lagrange pde satisfied by the minimizer is minimal surface equation

$$\sum_{i=1}^{n} \left(\frac{u_{x_i}}{(1+|Du|^2)^{1/2}} \right)_{x_i} = 0, \qquad \text{in } \Omega.$$

4. α-LAPLACIAN.

If $L(p, z, x) = |p|^\alpha$, $1 \leq \alpha < \infty$, then the functional I is

$$I[w] = \int_\Omega |Dw|^\alpha \, dx,$$

and the Euler-Lagrange pde satisfied by the minimizer is the α-Laplace's equation

$$\sum_{i=1}^{n} \left(|Du|^{\alpha-2} u_{x_i} \right)_{x_i} = 0, \qquad \text{in } \Omega.$$

Going back to the general case, and since $i(t)$ attains its minimum at $t = 0$, it follows that

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After similar calculations as before, this inequality gives the ellipticity condition:

$$\sum_{i,j=1}^{n} L_{p_i p_j}(Du, u, x)\, \xi_i\, \xi_j \geq 0, \qquad \text{for all } x \in \Omega \text{ and all } \xi \in \mathbb{R}^n.$$

This convexity condition is important to prove existence of solutions. **Exercise 3: see proof in Evans' book**

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- Suppose $\kappa(x) > 0$ is a function on a domain Ω.
- Find a surface u whose Gauss curvature at each point is $\kappa(x)$.
- The surface with this curvature solves the equation

$$\det D^2 u = \kappa(x)\,(1 + |Du|^2)^{(n+2)/2}.$$

This is an equation of Monge-Ampère type:

$$\det D^2 u = f(x, u, Du)$$

and is perhaps the most important fully nonlinear pde because of its applications and also is a model for other fully nonlinear pdes.

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$$\sigma_k(\lambda) = \sum_{1 \le i_1 \le \cdots \le i_n} \lambda_{i_1} \cdots \lambda_{i_k},$$

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If $k = 1$, then $\sigma_1\Big(\lambda(D^2u)\Big) = \lambda_1 + \cdots + \lambda_n = \Delta u$, that is, the Laplace equationand if $k = n$, then $\sigma_n\Big(\lambda(D^2u)\Big) = \lambda_1 \cdots \lambda_n = \det D^2u$, that is, the Monge-Ampère equation. Variants of these equations appear in conformal geometry.

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- A cost function is given:

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- Find a measure preserving map $t : \Omega_1 \to \Omega_2$, i.e., $|t^{-1}(E)| = |E|$ for each $E \subset \Omega_2$ such that minimizes the total cost

$$\int_{\Omega_1} c(x, t(x))\, dx$$

Theorem . $c : \mathbb{R}^n \to \mathbb{R}$ *strictly convex,* $f, g \in L^1(\mathbb{R}^n)$ *nonnegative with bounded support, and* $\displaystyle\int_{\mathbb{R}^n} f(x)\,dx = \int_{\mathbb{R}^n} g(y)\,dy$.

Theorem . $c : \mathbb{R}^n \to \mathbb{R}$ *strictly convex,* $f, g \in L^1(\mathbb{R}^n)$ *nonnegative with bounded support, and* $\int_{\mathbb{R}^n} f(x)\,dx = \int_{\mathbb{R}^n} g(y)\,dy$. *Given* $s \in \mathcal{S}(f,g)$, *i.e.,* $\int_{s^{-1}(E)} f(x)\,dx = \int_E g(y)\,dy$ *for each Borel set* $E \subset \mathbb{R}^n$, *let*

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Theorem . *$c : \mathbb{R}^n \to \mathbb{R}$ strictly convex, $f, g \in L^1(\mathbb{R}^n)$ nonnegative with bounded support, and $\int_{\mathbb{R}^n} f(x)\,dx = \int_{\mathbb{R}^n} g(y)\,dy$. Given $s \in \mathcal{S}(f,g)$, i.e., $\int_{s^{-1}(E)} f(x)\,dx = \int_E g(y)\,dy$ for each Borel set $E \subset \mathbb{R}^n$, let*

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Then

1. *$\exists$ unique $t \in \mathcal{S}(f,g)$ 1-to-1 such that $C(t) = \inf_{s\in\mathcal{S}(f,g)} C(s)$;*

2. *$\exists$ a c-convex function u such that*

$$t(x) = x - (Dc)^{-1}(-Du(x)) \text{ a.e.}$$

The pde associated with the problem is

$$g(x - Dc^*(-Du(x)))\,\det[I + D^2c^*(-Du(x))D^2u(x)] = f(x) \text{ in } \Omega,$$

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If $c(x) = \frac{1}{2}|x|^2$, then $c^*(x) = \frac{1}{2}|x|^2$ and the equation becomes

$$g(x + Du(x))\,\det[I + D^2u(x)] = f(x)$$

which is the M-A equation for $\frac{1}{2}|x|^2 + u(x)$.

II

FULLY NONLINEAR EE

Form of a nonlinear ee.

Let Ω be a domain in $\mathbb{R}^n$ and

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be continuous, where $\mathbb{S}^n$ denotes the real symmetric matrices.

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be continuous, where $\mathbb{S}^n$ denotes the real symmetric matrices.

A fully nonlinear pde is

$$F(x, u(x), Du(x), D^2u(x)) = 0, \qquad \text{in } \Omega, \tag{1}$$

and u is a classical solution if $u \in C^2(\Omega)$.

Suppose F is differentiable with respect to r at $\gamma = (x, z, p, r)$. If

$$F_r(\gamma) = \left[\frac{\partial F}{\partial r_{ij}}(\gamma)\right] > 0,$$

that is, the matrix $F_r(\gamma)$ is positive definite, then we say that F is elliptic at γ.

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In general, the equation (1) is elliptic on a function $u \in C^2(\Omega)$ if the matrix $\left[\frac{\partial F}{\partial r_{ij}}(x, Du(x), D^2u(x))\right]$ is positive definite at each $x \in \Omega$.

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For the Monge-Ampère equation $F(D^2u) = \det D^2u$ we have that $F_{r_{ij}}(D^2u(x)) = (D^2u)^{ij}(x)$ the ij-cofactor of $D^2u(x)$. Therefore, the M-A equation is elliptic in the class of convex functions.

III

NOTIONS OF SOLUTIONS

A main idea is to introduce a notion of solution that does not deal too much with the equation in a pointwise way and in such a way that is flexible and takes into account the basic structure of the equation.

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- Existence of weak solutions

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The introduction of the notion of weak solution to a pde is a very important methodological point that breaks the investigation of the pde into two parts:

- Existence of weak solutions
- Regularity of weak solutions

A. Weak solutions to divergence form equations

Divergent form pdes have a natural definition of solution: weak solution in the sense of integrating by parts against a test function.

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u is a weak solution to

$$-\sum_{i,j=1}^{n} \left(a_{ij}\, u_{x_j}\right)_{x_i} = f$$

if $u, Du \in L^2_{\text{loc}}(\Omega)$ and

$$\int_\Omega \sum_{i,j=1}^{n} a_{ij}(x)\, u_{x_i}(x)\, v_{x_j}(x)\, dx = \int_\Omega f(x)\, v(x)\, dx, \quad \forall v \in C_0^\infty(\Omega).$$

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Theorem . *Weak solutions to a linear uniformly elliptic pde with measurable coefficients are Hölder continuous with a modulus of continuity independent of the coefficients and depending only on the ellipticity constants.*

The technique use in the proof of this theorem is based on two important facts:

the Sobolev imbedding theorem: if $Du \in L^2$ then $u \in L^{2+\epsilon}$

$$||u||_{L^{2+\epsilon}(B)} \leq C\,||Du||_{L^2(B)}$$

and Caccioppoli's estimates: if u is a solution to the equation, then

$$\|Du\|_{L^2(B)} \leq C \|u\|_{L^2(2B)}.$$

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These two inequalities combined and applied to powers of the solution u yield the necessary L^∞-estimates of u that lead to Harnack's inequality:

There exists a constant $C > 0$ depending only on the ellipticity constants and dimension such that for each $u \geq 0$ weak solution of the equation, we have

$$\sup_B u \leq C \, \inf_B u$$

for all balls B such that $2B \subset \Omega$.

B. Viscosity solutions

A function $u \in C(\Omega)$ is a viscosity subsolution (supersolution) to (1) if whenever $y \in \Omega$ and $\phi \in C^2(\Omega)$ are such that

$$u - \phi \text{ has a local maximum (minimum) at } y$$

then

$$F(y, \phi(y), D\phi(y), D^2\phi(y)) \geq \; (\leq)\, 0.$$

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This notion was introduced by Crandall and Lions for first order eqs and later extended for second order eqs by Jensen.

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This notion was introduced by Crandall and Lions for first order eqs and later extended for second order eqs by Jensen.Viscosity solutions are useful proving existence using the Perron method (using the maximum principle), (Ishii).

B. Viscosity solutions

A function $u \in C(\Omega)$ is a viscosity subsolution (supersolution) to (1) if whenever $y \in \Omega$ and $\phi \in C^2(\Omega)$ are such that

$u - \phi$ has a local maximum (minimum) at y

then

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This notion was introduced by Crandall and Lions for first order eqs and later extended for second order eqs by Jensen.Viscosity solutions are useful proving existence using the Perron method (using the maximum principle), (Ishii).If F is convex or concave and uniformly elliptic, then viscosity solutions are $C^{2,\alpha}$, (Cabré and Caffarelli).

For elliptic linear equations in non divergence form we have the Harnack inequality of Krylov and Safonov:

it $u \geq 0$ is a solution to the elliptic non divergence form pde $a_{ij}D_{ij}u = 0$ in Ω, then

$$\sup_B u \leq C \inf_B u$$

for all balls B with $2B \subset \Omega$ with C independent of u and depending only of the ellipticity constants and dimension.

Non divergence equations appear are natural in diffusion processes and optimal control.

C. Aleksandrov solutions for the M-A equation

- Monge-Ampère equation:

$$\det D^2 u(x) = f(x).$$

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- **Normal or gradient map** of $u : \Omega \to \mathbb{R}$, $\Omega \subset \mathbb{R}^n$

$$\partial u(y) = \{p \in \mathbb{R}^n : u(x) \geq u(y) + p \cdot (x - y), \forall x \in \Omega\}$$

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$$Mu(E) = |\partial u(E)|.$$

- If $u \in C^2(\Omega)$ and convex, then

$$Mu(E) = \int_E \det D^2 u(x)\, dx.$$

- Let ν be a Borel measure on Ω. $u \in C(\Omega)$, convex, is a **weak solution** (Aleksandrov) if

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  For Monge-Ampère the ellipticity in linear equations is replaced by

$$\lambda\, |E| \le Mu(E) \le \Lambda\, |E|,$$

  then we have a regularity theory for solutions of $Mu = f$. For example, Aleksandrov solutions are regular, $C^{1,\alpha}$.

Some features of the Monge-Ampère equation

- It is an extremal for all elliptic equations:

$$(\det A)^{1/n} = \inf\{\operatorname{trace}(BA) : B \text{ is symmetric and } \det B = 1\}.$$

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- The equation is invariant by affine transformation that have determinant one:

$$
\begin{aligned}
v(x) &= u(Ax) \\
D^2 v(y) &= A^t \left(\left(D^2 u\right)(Ay)\right) A \\
\det D^2 v(y) &= (\det A)^2 \det D^2 u(Ay).
\end{aligned}
$$

IV

Important techniques arising in NEE

A priori estimates: the method of continuity.

Boundary value problems for nonlinear elliptic equations were first investigated by Serge Bernstein. The main idea consists in proving sufficiently strong a priori estimates for the prospective solutions.

A priori estimates: the method of continuity.

Boundary value problems for nonlinear elliptic equations were first investigated by Serge Bernstein. The main idea consists in proving sufficiently strong a priori estimates for the prospective solutions.

J. Schauder proved a priori estimates for linear elliptic equations of the form

$$L = \sum_{i,j=1}^{n} a_{ij}(x)\partial_{ij} + \sum_{i=1}^{n} b_j(x)\partial_i + c$$

and used them to solve the boundary value problem for this equation. He used the method of continuity as a way to solve the Dirichlet problem for a general operator L. The continuity method consists in introducing a parameter $t \in [0, 1]$ connecting the equation under consideration with a simpler one.

Exercise 4: see proof and details in Gilbarg and Trudinger book

Schauder estimates are the following:

If Ω is a $C^{2,\alpha}$ domain, $L = \sum_{i,j=1}^{n} a_{ij}(x)\partial_{ij} + \sum_{i=1}^{n} b_j(x)\partial_i + c$ is a linear and uniformly elliptic differential operator with C^α coefficients, $f \in C^\alpha$ and $u \in C^{2,\alpha}(\bar{\Omega})$ is a solution to $Lu = f$ in Ω, $\phi \in C^{2,\alpha}(\bar{\Omega})$ such that $\phi = u$ on $\partial\Omega$, then u satisfies the apriori estimate

$$||u||_{C^{2,\alpha}(\Omega)} \leq C \left(||u||_{C^0(\Omega)} + ||\phi||_{C^{2,\alpha}(\Omega)} + ||f||_{C^0(\Omega)} \right),$$

where C depends only on the ellipticity constants, the bounds of the Hölder norms of the coefficients, n, and Ω.

Evans-Krylov a priori estimates.

$F(D^2u)$ uniformly elliptic, with F concave; $u \in C^2(B_1)$ solution to $F(D^2u) = 0$ in B_1. Then $u \in C^{2,\alpha}(B_{1/2})$ and

$$||u||_{C^{2,\alpha}(B_{1/2})} \leq C\,||u||_{C^{1,1}(B_{3/4})}$$

where $0 < \alpha < 1$ and C are universal constants.

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$$\|u\|_{C^{2,\alpha}(B_{1/2})} \leq C\,\|u\|_{C^{1,1}(B_{3/4})}$$

where $0 < \alpha < 1$ and C are universal constants.

It is an open problem to determine if these estimates still hold for non concave (or non convex) F.

Perturbation methods.

Using the Calderón-Zgmund theory of singular integrals in L^p spaces one obtains estimates of the second derivatives of elliptic equations with continuous coefficients. This uses the theory of singular integrals of C-Z.

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Precisely, if

$$L = \sum_{i,j=1}^{n} a_{ij}(x)\partial_{ij}$$

is a uniformly elliptic operator with continuous coefficients, the idea is the freeze the coefficient at a point and consider

$$L = \sum_{i,j=1}^{n} \left(a_{ij}(x) - a_{ij}(x_0)\right) \partial_{ij} + \sum_{i,j=1}^{n} a_{ij}(x_0)\partial_{ij}$$

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$$L = \sum_{i,j=1}^{n} \Big(a_{ij}(x) - a_{ij}(x_0) \Big) \partial_{ij} + \sum_{i,j=1}^{n} a_{ij}(x_0) \partial_{ij}$$

The operator $L_0 = \sum_{i,j=1}^{n} a_{ij}(x_0)\partial_{ij}$ has constant coefficients and by changing the coordinates is the Laplace operator and from the Calderón-Zygmund theory of singular integrals

$$\|D^2 u\|_{L^p} \leq C \, \|L_0 u\|_{L^p}.$$

Maximum principles.

Elliptic equations satisfy the maximum principle: if u is a solution to a linear elliptic pde $Lu = 0$ in Ω, then

$$\max_{\Omega} u = \max_{\partial\Omega} u.$$

Also, if u solves $Lu = f$ in Ω, then

$$\max_{\Omega} u \leq \max_{\partial\Omega} u + C\,\|f\|_{L^n(\Omega)},$$

where C is a constant depending only the ellipticity, Ω and the dimension. This last inequality follows from an estimate for the

Monge-Ampère operator. This dependence of C is important for nonlinear problems. **Exercise 5: see proof for L^∞ case in Gilbarg and Trudinger book**

Moreover, nonlinear elliptic equations satisfy what is called comparison principles: if $F(u) \le F(v)$ in some sense in Ω, then

$$\min_{\Omega}(u - v) = \min_{\partial\Omega}(u - v).$$

Estimates of second derivatives.

By means of paraboloids tangent to the graph of the solution one can estimate the size of the second derivatives.

Here the notion of viscosity solution is useful, the estimates of second derivatives are done estimating the opening of the touching paraboloid.

A-B-P and maximum principle are used to estimate the distribution function of solutions which together with the Calderón-Zygmund decomposition in cubes yields to Harnack's inequality.

V

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