Topics in Monge-Ampère type Equations II

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PLAN OF THE TALK

I Basic facts for M-A

II Notion of weak solution

III Solvability of the Dirichlet problem

IV Regularity of solutions

V A M-A type equation in the construction of reflectors

I

Basic facts for M-A

- Monge-Ampère equation:

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For Monge-Ampère $F_{r_{ij}}(D^2u(x)) = (D^2u)^{ij}(x)$ the ij-cofactor of $D^2u(x)$. Therefore, the M-A equation is elliptic in the class of convex functions.

- It is an extremal for all elliptic equations:

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- The equation is invariant by affine transformations with determinant one:

$$\begin{aligned} v(x) &= u(Ax) \\ D^2 v(y) &= A^t \left(\left(D^2 u\right)(Ay)\right) A \\ \det D^2 v(y) &= (\det A)^2 \, \det D^2 u(Ay). \end{aligned}$$

II

Weak solutions

- **Normal or gradient map** of $u : \Omega \to \mathbb{R}$, $\Omega \subset \mathbb{R}^n$

$$\partial u(y) = \{p \in \mathbb{R}^n : u(x) \geq u(y) + p \cdot (x - y), \forall x \in \Omega\}$$

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- Legendre transform: $u^*(y) = \sup_{x \in \Omega} x \cdot y - u(x)$; always convex so differentiable a.e.

- If $u \in C(\bar{\Omega})$, then the set

$$F = \{p \in \mathbb{R}^n : p \in \partial u(x) \cap \partial u(y), x \neq y\}$$

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Two important results

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- Basic property: $v = u$ on $\partial\Omega$, $v \geq u$ in $\Omega \Longrightarrow \partial v(\Omega) \subset \partial u(\Omega)$.

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Theorem. [Aleksandrov maximum principle] *Ω convex bounded, $u \in C(\bar{\Omega})$, $u = 0$ on $\partial\Omega$, then*

$$|u(y)|^n \leq C_n \, \mathit{diam}(\Omega)^{n-1} \, \mathit{dist}(y, \partial\Omega) \, Mu(\Omega).$$

Proof.

- v is the cone with vertex $(y, u(y))$ and base $\Omega \times \{0\}$.

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- *convex hull* $\left\{ B\left(0, \frac{-u(y)}{\text{diam}(\Omega)}\right), p \right\} \subset \partial v(\Omega) \subset \partial u(\Omega)$,

  with $|p| = -\frac{u(y)}{\text{dist}(y, \partial\Omega)}$.

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Then we get Aleksandrov. $\square$

Theorem. [Comparison principle] $u, v \in C(\bar{\Omega})$ *with*

$$Mu \leq Mv.$$

Then

$$\min_{\bar{\Omega}}(u - v) = \min_{\partial\Omega}(u - v).$$

III

Solvability of the Dirichlet problem

Homogeneous Dirichlet problem

Theorem. *Ω strictly convex, $g \in C(\partial\Omega)$. Then $\exists!$ $u \in C(\overline{\Omega})$ convex weak solution to*

$$Mu = 0$$
$$u = g \qquad \text{on } \partial\Omega.$$

Proof.

$$\mathcal{F} = \{a(x) \text{ affine with } a \leq g \text{ on } \partial\Omega\}$$
$$u(x) = \sup\{a(x) : a \in \mathcal{F}\}$$

Ω strictly convex $\Longrightarrow u = g$ on $\partial\Omega$.

The linear nature of $u \Longrightarrow$

$\partial u(\Omega) \subset \{p \in \mathbb{R}^n : p \in \partial u(x) \cap \partial u(y), x \neq y\}$. $\square$

Non Homogeneous Dirichlet problem

Theorem. *Ω strictly convex, $\mu(\Omega) < \infty$, $g \in C(\partial\Omega)$. Then $\exists!$ $u \in C(\overline{\Omega})$ convex weak solution to*

$$Mu = \mu$$

$$u = g \qquad \text{on } \partial\Omega.$$

Proof. Based in the Perron method:

$$\mathcal{F}(\mu, g) = \{v \in C(\bar{\Omega}) : v \text{ convex}, Mv \geq \mu, v = g \text{ on } \partial\Omega\},$$

Let W solve $MW = 0$, $W = g$ on $\partial\Omega$.

Then $MW \leq \mu \leq Mv$ for $v \in \mathcal{F}(\mu, g)$ and so by the comparison principle

$$v \leq W.$$

If $\mathcal{F}(\mu, g) \neq \emptyset$, then let

$$U(x) = \sup\{v(x) : v \in \mathcal{F}(\mu, g)\}.$$

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This is the solution.To prove this we proceed by approximation: approximate μ by a sequence of measures of the form $\mu_N = \sum_{i=1}^{N} a_i \, \delta_{x_i}$ with $a_i > 0$ and $x_i \in \Omega$.

- $\mathcal{F}(\mu_N, g) \neq \emptyset$. Notice that, $M(|x - x_i|) = \omega_n \, \delta_{x_i}$.

  Let $f(x) = \frac{1}{\omega_n^{1/n}} \sum_{i=1}^{N} a_i^{1/n} |x - x_i|$.

Solve $Mu = 0$ in Ω with $u = g - f$ on $\partial\Omega$. The function $v = u + f$ belongs to $\mathcal{F}(\mu_N, g)$, because $Mv \geq Mu + Mf$.

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- $MU \leq \mu$. Define the "lifting" of U, solve $Mv = 0$ in $B_r(x_0)$ with $v = U$ on $\partial B_r(x_0)$ and let

$$w(x) = \begin{cases} v(x) \text{ for } x \in B_r(x_0) \\ U(x) \text{ for } x \in B_r(x_0)^C \end{cases}$$

□

IV

Regularity of solutions

- If $L = D_i(a_{ij}(x)D_j)$ or $L = a_{ij}(x)D_{ij}$ satisfies

$$\lambda\, Id \leq (a_{ij}(x)) \leq \Lambda\, Id,$$

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For Monge-Ampère the ellipticity in linear equations is replaced by

$$\lambda\, |E| \leq Mu(E) \leq \Lambda\, |E|,$$

then we have a regularity theory for solutions of $Mu = f$. For example, Aleksandrov solutions are regular (assuming appropriate bdry data): $C^{1,\alpha}$.

- Ω convex is normalized if $B_{\alpha_n}(0) \subset \Omega \subset B_1(0)$.
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- Ω open, bounded and convex; $\exists\ T$ affine such that $T(\Omega)$ is normalized.
- The **sections** of u are the convex sets

$$S(y,t) =$$

$$\{x \in \Omega : u(x) < u(y) + Du(y) \cdot (x - y) + t\}.$$

Geometric properties of sections

Assume $\lambda \leq \det D^2 u \leq \Lambda$, $u = 0$ on $\partial\Omega$.

Sections behave like Euclidean balls after affine transformations.

- Let T be affine normalizing $S_u(x,t)$, $T(S_u(x,t)) = S_v(Tx,t)$ where $v(y) = u(T^{-1}y)$.

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- Engulfing property: There exists a constant $\vartheta > 0$ such that

$$y \in S(z,t) \Longrightarrow S(z,t) \subset S(y, \vartheta t).$$

- Let

$$\Omega_\alpha = \{x \in \Omega : u(x) < (1-\alpha) \min_{\Omega} u\}.$$

Given $0 < \alpha < 1$, there exists $\eta(\alpha) > 0$ such that if $x \in \Omega_\alpha$ and $t \leq \eta(\alpha)$, then $S(x,t) \Subset \Omega$.

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- There exist positive constants K_1, K_2, K_3 and ϵ such that if $S(z_0, r_0)$ and $S(z_1, r_1)$ are sections with $r_1 \leq r_0$, $S(z_0, r_0) \cap S(z_1, r_1) \neq \emptyset$ and T is an affine transformation that normalizes $S(z_0, r_0)$ then
$$B\left(Tz_1, K_2 \frac{r_1}{r_0}\right) \subset T(S(z_1, r_1)) \subset B\left(Tz_1, K_1 \left(\frac{r_1}{r_0}\right)^\epsilon\right),$$
and $Tz_1 \in B(0, K_3)$.

- Sections are strictly convex.

These geometric properties permit to establish estimates of solutions using measure theoretical arguments. Mainly covering lemmas which yield distribution function estimates (Harmonic analysis techniques).

I will describe some known results for M-A.

We assume Ω strictly convex normalized domain,
$\det D^2 u(x) = f(x)$, $u = 0$ on $\partial\Omega$.

Second derivatives estimates for M-A

- $f = 1$, then L^∞-estimates (Pogorelov, 71):

$$C_1\, Id \leq D^2 u(x) \leq C_2\, Id, \quad x \in \Omega'$$

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- f continuous, $\lambda \le f \le \Lambda$, then L^p interior estimates (Caffarelli, 90):

$$\|D^2 u\|_{L^p(\Omega')} \le C,$$

$\Omega' \Subset \Omega$, and C universal positive constant. If $f \in C^\alpha$ then $u \in C^{2,\alpha}$.

- w_f satisfies a Dini condition, then $D^2u \in L^\infty(\Omega')$, and $\exists\ f \in C(\Omega)$, with $D^2u \notin L^\infty$, (X-J. Wang, 95).

Estimates for the linearized M-A equation

- $u : \mathbb{R}^n \to \mathbb{R}$,

$$\det D^2 (u + tv) =$$
$$\det D^2 u + t \operatorname{trace}\left(U D^2 v\right) + \cdots + t^n \det D^2 v,$$

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$U = (U)_{ij}$ is the matrix of cofactors of $D^2 u$.

- **The linearized Monge-Ampère operator is:**

$$L_U v = \operatorname{trace}\left(U D^2 v\right).$$

- $L_U u = n \det D^2 u$; div $(U D v) = \operatorname{trace}(U D^2 v)$.

- D^2u is positive semidefinite $\Rightarrow U$ is positive semidefinite. Thus, L_U is elliptic, possibly degenerate.

- L_U appears when one considers first derivatives of u: $u_j = D_j u$, $j = 1, \ldots, n$, satisfy the equation

$$L_U(u_j) = \mu_j.$$

Theorem . [L. A. Caffarelli & C.G.] *Suppose*

$$\lambda \leq \det D^2 u \leq \Lambda.$$

There exists a constant C depending only on n, λ and Λ such that

$$\sup_S v \leq C \inf_S v,$$

for all sections S of u and for each solution $v \geq 0$ to $L_U v = 0$.

To put the results of the next part in perspective we mention the following result.

Theorem. [Caffarelli] *Suppose u convex in Ω is a weak solution to the Monge-Ampère equation*

$$\lambda \leq \det D^2 u \leq \Lambda.$$

If ℓ is a supporting hyperplane to u, and the set $\Gamma = \{x : u(x) = \ell(x)\}$ has more than one point, then Γ has no extremal points in the interior of Ω.

This implies that solutions to M-A with $u = 0$ on $\partial\Omega$ are strictly convex.

V

A M-A TYPE EQUATION

IN THE CONSTRUCTION OF REFLECTORS

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- Snell's law of reflection on a perfectly reflecting surface S:

  angle(normal to tangent plane, incident ray)

  $=$ angle(normal to tangent plane, reflected ray).

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- Paraboloid: rays emanating from the focus are reflected into parallel rays.

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If $\Omega \subset S^{n-1}$ and a ray emanates from O in the direction $x \in \Omega$, and $\mathcal{A}$ is a reflecting surface parameterized by

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If $\Omega \subset S^{n-1}$ and a ray emanates from O in the direction $x \in \Omega$, and $\mathcal{A}$ is a reflecting surface parameterized by

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then the reflected ray has direction:

$$T(x) = x - 2\langle x, \nu\rangle \nu,$$

where ν is the outer normal to $\mathcal{A}$ at $z = x\rho(x)$.

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Light emanates from O with intensity $f(x)$ for $x \in \Omega$.

Find a reflecting surface $\mathcal{A}$ parameterized by $z = x\rho(x)$ for $x \in \Omega$, such that all reflected rays by $\mathcal{A}$ fall in the direction Ω^*, and the output illumination received in the direction x^* is $g(x^*)$.

- No loss of energy in the reflection $\Longrightarrow$ by conservation of energy that

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- In addition, and again by conservation of energy, T is a measure preserving map:

$$\int_{T^{-1}(E)} f(x)\, dx = \int_{E} g(x)\, dx, \qquad \text{for all } E \subset \Omega^* \text{ Borel set}$$

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Given Ω, Ω^* domains on S^{n-1}, $f \in C(\Omega)$, $g \in C(\Omega^*)$ positive functions, find a reflecting surface $\mathcal{A}$ parameterized by $z = x\rho(x)$ for $x \in \Omega$ such that

$$T(\Omega) = \Omega^*.$$

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  Taking an orthonormal frame of coordinates on S^{n-1} yields the pde

$$\mathcal{L}\rho = \eta^{-2}\det\left(-\nabla_i\nabla_j\rho + 2\rho^{-1}\nabla_i\rho\nabla_j\rho + (\rho-\eta)\delta_{ij}\right) = \frac{f(x)}{g(T(x))};$$

  where ∇=covariant derivative, $\eta = \dfrac{|\nabla\rho|^2+\rho^2}{2\rho}$, and δ_{ij} is the Kroeneker delta.

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Very complicated fully nonlinear pde of Monge-Ampère type.

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- A connection with mass transportation in the sphere was found by Xu-Jia Wang (2003), the cost function is $c(x, y) = -\log(1 - x \cdot y)$. Also a connection with mass transportation was found in the case of two reflectors by T. Glimm and V. Oliker (2002).

V.A

NOTION OF WEAK SOLUTION

SUPPORTING PARABOLOIDS

- Let $m \in S^{n-1}$, and $b > 0$. $P(m, b)$ denotes the paraboloid in $\mathbb{R}^n$ with focus at 0, axis m, and directrix plane $m \cdot x + 2b = 0$.

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- Given $\rho(x) \in C(S^{n-1})$, $\rho > 0$, the radial antenna associated with ρ is

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- The antenna $\mathcal{A}$ is an admissible surface if it has a supporting paraboloid at each point.

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- The class of sets $E \subset S^{n-1}$ for which $\mathcal{N}_{\mathcal{A}}(E)$ is Lebesgue measurable is a Borel σ-algebra.

WEAK SOLUTIONS

- Given $g \in L^1(S^{n-1})$ we define the Borel measure

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- Smooth solutions to $\mathcal{L}\rho = f/g$ are weak solutions.

PROGRAM

How regular are weak solutions?

are they ”strictly convex”?

$C^{1,\alpha}$?

$C^{2,\alpha}$?

V.B

RESULTS

Theorem. [Caffarelli, Gutiérrez and Q. Huang, Ann. of Math. to appea

Suppose $\mathcal{A}$ *is an antenna given by* $\mathcal{A} = \{x\rho(x) : x \in S^{n-1}\}$, *and satisfying*

1. $\exists$ *constants* r_1, r_2 *such that* $0 < r_1 \le \rho(x) \le r_2$ *for all* $x \in S^{n-1}$.

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If $P(m, a)$ *is a supporting paraboloid to* $\mathcal{A}$, *then* $P(m, a) \cap \mathcal{A}$ *is a single point set. In addition,* $\mathcal{A}$ *is a* C^1 *surface.*

V.C

IDEA OF THE PROOF

- $m \in S^{n-1}$, $b > 0$, $P(m, b)$ is the paraboloid in $\mathbb{R}^n$ with focus at 0, axis m, and directrix hyperplane $\Pi(m, b)$ of equation $m \cdot y + 2b = 0$. $P(m, b)$ has equation

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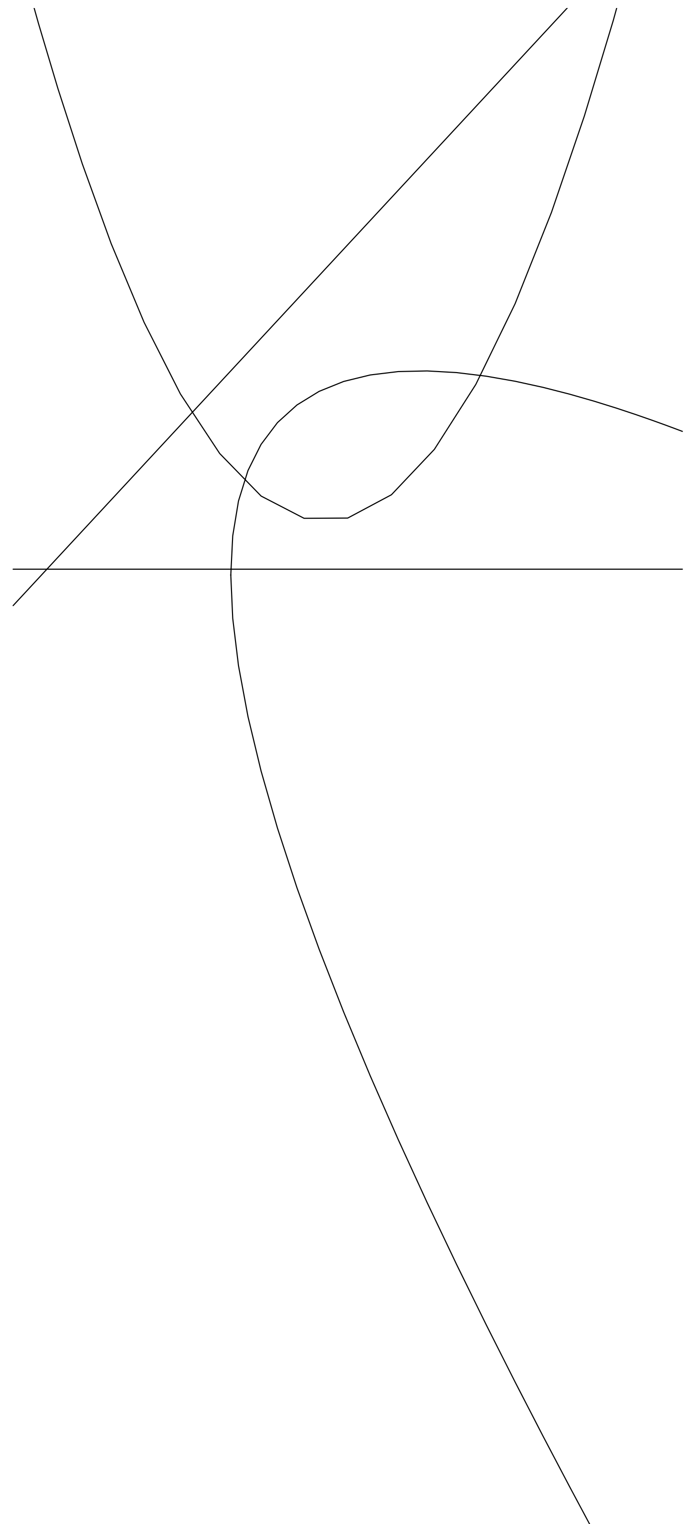
- IMPORTANT SIMPLE FACT: if $P(m', b')$ is another paraboloid, then $P(m, b) \cap P(m', b')$ is contained in the bisector of the directrices of both paraboloids, denoted by $\Pi[(m, b),\ (m', b')]$ and whose equation is

$$\Pi[(m, b),\ (m', b')] \equiv \{y : (m - m') \cdot y + 2(b - b') = 0\}.$$



UPPER ESTIMATE

Consider the portion $\mathcal{R}$ of the antenna $\mathcal{A}$ between the paraboloids $P(e_n, a)$ and $P(e_n, a + h)$ with $h > 0$ which is a supporting paraboloid to $\mathcal{A}$, and where e_n is the nth-coordinate vector in $\mathbb{R}^n$.

Suppose that $P(m, b)$ is a supporting paraboloid to $\mathcal{A}$ at some point y_0 in the portion $\mathcal{R}$.

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Precisely:

Proposition 1. *Let E be the ellipsoid of minimum volume containing the projection over $\mathbb{R}^{n-1}$ of $\mathcal{R}$.*

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If $\mathcal{R}_{1/2}$ is the portion of $\mathcal{R}$ whose projection on $\mathbb{R}^{n-1}$ is the contracted ellipsoid $(1/2(n-1))E$,then

$$\mathcal{N}_{\mathcal{A}}(\mathcal{R}_{1/2}) \subset$$

$$\{m \in S^{n-1}; |m_i| \leq C\, h/\lambda_i, \quad i = 1, \cdots, n-1,$$

$$\text{and } |m'| \leq \sqrt{2(1-m_n)} \leq C\sqrt{h}/\operatorname{diam}(E)\}.$$

LOWER ESTIMATE: ALEKSANDROV TYPE

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$$\begin{aligned} \{m \in S^{n-1} : \sqrt{1 - m_n} \leq \varepsilon_0 \frac{\sqrt{h}}{\operatorname{diam}(E)}, \\ 0 \leq -m_1 \leq \varepsilon_0 \frac{h}{\delta_z \lambda_1}, |m_i| \leq \varepsilon_0 \frac{h}{\lambda_i} \text{ for } i = 2, \cdots, n-1\} \\ \subset \mathcal{N}_{\mathcal{A}}(\mathcal{R}). \end{aligned}$$

PROOF OF THE THEOREM

$$\Delta = \text{ projection of } P(e_n, a_1) \cap \mathcal{A} \text{ on } \mathbb{R}^{n-1}$$

If Δ contains at least two points, then $\mathrm{diam}(\Delta) = C$.

$$\mathcal{R}_h = \text{ portion of } \mathcal{A} \text{ cut by } P(e_n, a_1 - h)$$

$$\mathcal{R}'_h = \text{ projection of } \mathcal{R}_h \text{ on } \mathbb{R}^{n-1}$$

$$E_h = \text{ John's ellipsoid for } \mathcal{R}'_h$$

$$(\mathcal{R}_h)_{1/2} = \text{ lower portion of } \mathcal{R}_h \text{ over } \frac{1}{2(n-1)} E_h$$

$$D_h = \text{ preimage of } \mathcal{R}_h \text{ in } S^{n-1}$$

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We have that $\mathcal{R}_h \to \mathcal{R}_0$ in the Hausdorff metric and so $\mathcal{R}'_h \to \Delta$ in the same metric.

Let $\lambda_1(h)$ be the longest axis of E_h, then $\lambda_1(h) \approx \operatorname{diam}(\Delta)$. There exists $z_h \in \Delta$ such that $K - \delta_h \lambda_1(h) \leq (z_h)_1 \leq K = \sup_{z \in \mathcal{R}'_h} z_1$, and $\delta_h \to 0$.

We have

$$|D_h| \approx |\mathcal{R}'_h|$$

$$|(D_h)_{1/2}| \approx |(\mathcal{R}_h)_{1/2}| \approx |E_h|.$$

Then from Proposition 1,

$$C\,|E_h| \leq |\mathcal{N}((\mathcal{R}_h)_{1/2})| \leq \min\left\{\frac{C\,h}{\lambda_1}, \frac{C\;\sqrt{h}}{\operatorname{diam}(E_h)}\right\}$$

$$\prod_{i=2}^{n-1} \min\left\{\frac{C\,h}{\lambda_i}, \frac{C\;\sqrt{h}}{\operatorname{diam}(E_h)}\right\},$$

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and from Proposition 2

$$|E_h| \ge |\mathcal{R}'_h| \ge C\,|\mathcal{N}_{\mathcal{A}}(D_h)| \ge C \min\left\{\frac{\varepsilon_0\,h}{\delta_h\,\lambda_1}, \frac{\varepsilon_0\,\sqrt{h}}{\operatorname{diam}(E_h)}\right\}$$
$$\prod_{i=2}^{n-1} \min\left\{\frac{\varepsilon_0\,h}{\lambda_i}, \frac{\varepsilon_0\,\sqrt{h}}{\operatorname{diam}(E_h)}\right\}.$$

Therefore,

$$\varepsilon_0^{n-1}\,\min\left\{\frac{h}{\delta_h\,\lambda_1},\frac{\sqrt{h}}{\operatorname{diam}(E_h)}\right\}\le C\,\min\left\{\frac{h}{\lambda_1},\frac{\sqrt{h}}{\operatorname{diam}(E_h)}\right\}.$$

Since $\lambda_1 \approx \operatorname{diam}(E_h) \approx const$, we obtain the following contradiction

$$\varepsilon_0^{n-1}\,\min\left\{\frac{h}{\delta_h},\ \sqrt{h}\right\}\le C\,h,$$

for any $h > 0$ sufficiently small.