6. Characterization of Gabor systems that are Parseval frames

$$\mathcal{G}_{B,C}(g) = \{T_{Ck} M_{Bm} g : k, m \in \mathbb{Z}^n\}$$

Gabor system

In this case: $\mathcal{P} = \mathbb{Z}^n$; $C_p = C_m = C$ $\forall m \in \mathbb{Z}^n$ & $g_m = M_{Bm} g$.

$\Delta = \bigcup_{m \in \mathbb{Z}^n} C^I(\mathbb{Z}^n) = C^I(\mathbb{Z}^n)$. Thus, if $\alpha \in \Delta$, $\alpha = C^I m$ for some $m \in \mathbb{Z}^n$ and $\mathcal{P}_\alpha = \{p \in \mathcal{P} : C^t \alpha \in \mathbb{Z}^n\} = \mathbb{Z}^n$.

If $\mathcal{G}_{B,C}(g)$ satisfies L.I.C, by theorem 5.2, $\mathcal{G}_{B,C}(g)$ is a PF $\Leftrightarrow$

$$\sum_{k \in \mathbb{Z}^n}{}' \frac{1}{|\det C|} (M_{Bk} g)^\wedge(\xi) \overline{(M_{Bk} g)^\wedge(\xi + C^I m)} = \delta_{m,0} \quad \text{a.e. } \xi \in \mathbb{R}^n$$

for all $m \in \mathbb{Z}^n$. But $(M_{Bk} g)^\wedge = T_{Bk} \hat{g}$ so that

THEOREM 6.1.

> $\mathcal{G}_{B,C}(g)$ is a Parseval frame iff
>
> $$\sum_{k \in \mathbb{Z}^n} \frac{1}{|\det C|} \hat{g}(\xi - Bk) \overline{\hat{g}(\xi - Bk + C^I m)} = \delta_{m,0} \quad \text{a.e. } \xi \in \mathbb{R}^n$$
>
> for all $m \in \mathbb{Z}^n$.

P/ We need to prove that L.I.C is satisfied, that is

$$L(f) = \sum_{p \in \mathbb{Z}^n}{}' \frac{1}{|\det C|} \sum_{m \in \mathbb{Z}^n} \int_{\operatorname{sop} \hat{f}} |\hat{f}(\xi + C^I m)|^2 |(M_{Bp} g)^\wedge(\xi)|^2 \, d\xi < \infty$$

for all $f \in \mathcal{D}$. Since $(M_{Bp} g)^\wedge = T_{Bp} \hat{g}$ and $\hat{f}(\xi + C^I m)$ is non-zero for a finite number of m's, it is enough to prove the integrability of

$$\sum_{p\in\mathbb{Z}^n}' |\hat{f}(\xi + C^I m)|^2 |\hat{g}(\xi - Bp)|^2$$

for any compact subset $K \subset \mathbb{R}^n$ and all $m \in \mathbb{Z}^n$. Since $\|\hat{f}\|_\infty < \infty$, it is enough to prove

$$\int_K \sum_{p\in\mathbb{Z}^n}' |\hat{g}(\xi - Bp)|^2 < \infty \qquad (1)$$

for any compact subset $K \subset \mathbb{R}^n$. For each $j \in \mathbb{Z}^n$, $\{B(\mathbb{T}^n + j - p) : p \in \mathbb{Z}^n\}$ is a partition of $\mathbb{R}^n$, so that

$$\|g\|_2^2 = \int_{\bigcup\limits_{p\in\mathbb{Z}^n} B(\mathbb{T}^n + j - p)} |\hat{g}(\eta)|^2 d\eta = \int_{B(\mathbb{T}^n + j)} \sum_{p\in\mathbb{Z}^n}' |\hat{g}(\eta - Bp)|^2 d\eta.$$

This proves (1) since K is contained in a finite union of sets of the form $B(\mathbb{T}^n + j)$. ∎

There is a characterization of Gabor systems that are PF's that does not involve $\hat{g}$ but only g.

By Parseval / Plancherel

$$\mathcal{G}_{B,C}(g) = \{T_{Ck} M_{Bm} g : k, m \in \mathbb{Z}^n\} \text{ is a PF for } L^2(\mathbb{R}^n)$$

$$\Longleftrightarrow$$

$$\{M_{-Ck} T_{Bm} \hat{g} : k, m \in \mathbb{Z}^n\} \text{ is a PF for } L^2(\mathbb{R}^n)$$

But $M_{Ck} T_{Bm} \hat{g} = e^{-2\pi i Ck \cdot Bm} T_{Bm} M_{-Ck} \hat{g}$ and the factor $e^{-2\pi i Ck \cdot Bm}$ has $|\ \ | = 1$. Thus

$\mathcal{G}_{B,C}(g)$ is a PF $\Longleftrightarrow$

$$\mathcal{G}_{-C,B}(\hat{g}) = \{T_{Bm} M_{-Ck} \hat{g} : m, k \in \mathbb{Z}^n\} \text{ is a PF}$$

Use theorem 6.1 to obtain:

$\mathcal{G}_{B,C}(g)$ is a PF $\Leftrightarrow$

$$\sum_{m\in\mathbb{Z}^n} \frac{1}{|\det B|}\, g(\xi + Cm)\, \overline{g(\xi + Cm + B^T k)} = \delta_{k,0} \quad \text{a.e. } \xi\in\mathbb{R}^n \tag{2}$$

for all $k\in\mathbb{Z}^n$

This characterization can be used to find Gabor frames easily ("Painless non-orthogonal expansions as in [Daubechies, Grossman, Meyer]):

Take $B=\beta I_n$, $C=\gamma I_n$ so that (2) becomes

$$\sum_{m\in\mathbb{Z}^n} g(\xi-\gamma m)\,\overline{g(\xi-\gamma m + \tfrac{k}{\beta})} = \beta\, \delta_{k,0} \quad \text{a.e. } \xi\in\mathbb{R}^n \tag{3}$$

for all $k\in\mathbb{Z}^n$. Take $g\in L^2(\mathbb{R}^n)$ s.t. $\text{sop } g \subset [-\frac{1}{2\beta},\frac{1}{2\beta}]$. Hence, if $k\neq 0$ eq. (3) is trivially satisfied and we only need to find g so that

$$\sum_{m\in\mathbb{Z}^n} |g(\xi-\gamma m)|^2 = \beta \quad \text{a.e. } \xi\in\mathbb{R}^n \tag{4}$$

Eq. (4) can be satisfied by taking $g=\beta\, \chi_{[-\frac{\gamma}{2},\frac{\gamma}{2})^n}$ where we must have $\gamma \le \frac{1}{\beta}$. The function g is localized, but its Fourier transform

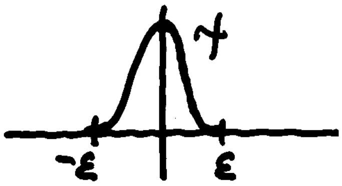
$$\hat{g}(\xi) = \beta \int_{[-\frac{\gamma}{2},\frac{\gamma}{2}]^n} e^{-2\pi i x.\xi}\,dx = \beta \prod_{j=1}^{n} \frac{\operatorname{sen} \pi i \gamma \xi_j}{\pi i \xi_j}$$

does not belong to $L^1(\mathbb{R}^n)$.

Gabor "atoms" g can be found so that $g \in \mathcal{S}$ (and hence $\hat{g} \in \mathcal{S}$) by carefully smoothing of the above example.

$n=1$; $\psi \in C^{\infty}(\mathbb{R})$, ψ even, $\operatorname{sop} \psi \subset [-\varepsilon, \varepsilon]$ & $\int_{-\varepsilon}^{\varepsilon} \psi(t)\,dt = \frac{\pi}{2}$

Write $\theta(x) = \int_{-\infty}^{x} \psi(t)\,dt$. Then



$$\theta(x) + \theta(-x) = \int_{-\infty}^{x} \psi(t)\,dt + \int_{-\infty}^{-x} \psi(t)\,dt$$

$$= \int_{-\infty}^{x} \psi(t)\,dt + \int_{x}^{\infty} \psi(-t)\,dt \overset{\psi \text{ even}}{=\!=\!=} \int_{-\infty}^{x} \psi(t)\,dt + \int_{x}^{\infty} \psi(t)\,dt = \frac{\pi}{2}$$

Write $S(x) \equiv S_{\varepsilon}(x) = \operatorname{sen} \theta(x)$ and $C(x) \equiv C_{\varepsilon}(x) = \cos \theta(x)$, so that

$$C(x) = \cos \theta(x) = \cos\left[\frac{\pi}{2} - \theta(-x)\right] = \operatorname{sen} \theta(-x) = S(-x)$$

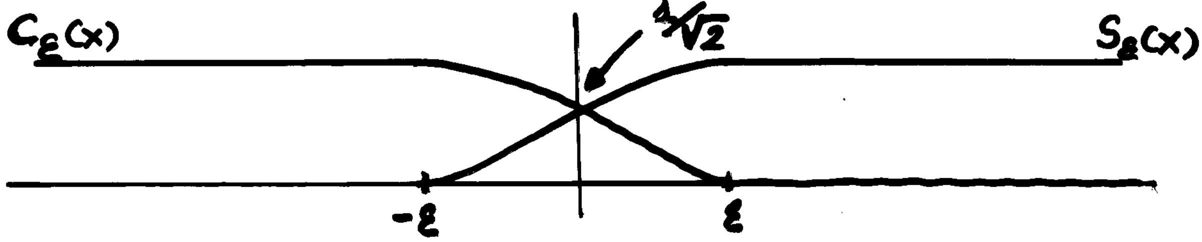
and

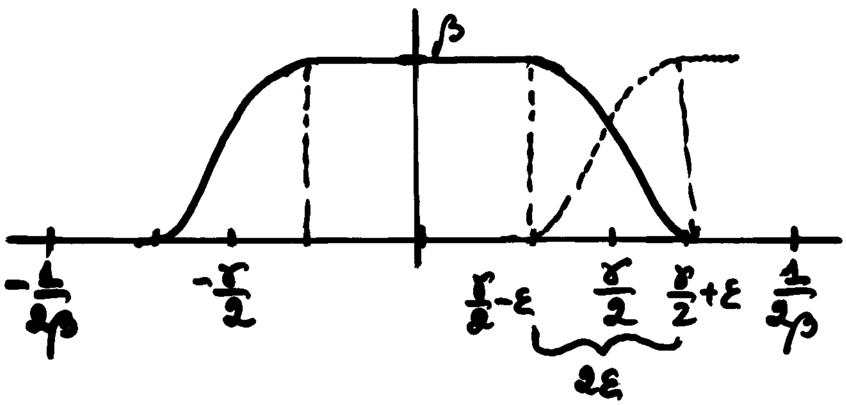
$$S^2(x) + C^2(x) = 1$$

Take $\gamma < \frac{1}{\beta}$ and ε s.t. $\frac{\gamma}{2} + \varepsilon < \frac{1}{2\beta}$ and $\varepsilon \leq \frac{\gamma}{2}$. Consider

$$g(x) = \sqrt{\beta}\, S_{\varepsilon}\left(x + \frac{\gamma}{2}\right) C_{\varepsilon}\left(x - \frac{\gamma}{2}\right)$$

to have the Gabor "atom" that satisfies (2).





Remark:

$$g^2(x-\gamma) + g^2(x) =$$
$$= \beta S_{\varepsilon}^2\left(x - \frac{\gamma}{2}\right) + \beta C_{\varepsilon}^2\left(x - \frac{\gamma}{2}\right)$$
$$= \beta \quad \text{if } x \in \left[\frac{\gamma}{2} - \varepsilon, \frac{\gamma}{2} + \varepsilon\right)$$

7. Wexler-Raz and Rieffel's Theorems for Gabor Frames

Let $\mathcal{G}_{B,C}(g)$ be a Gabor system,

$$\mathcal{G}_{B,C}(g) = \{T_{Ck} M_{Bm} g : k, m \in \mathbb{Z}^n\}.$$

The <u>adjoint</u> Gabor system is defined as

$$\mathcal{G}_{C^I, B^I}(\tilde{g}) = \{T_{B^I k} M_{C^I m} \tilde{g} : k, m \in \mathbb{Z}^n\}$$

where $C^I = (C^t)^{-1}$ and $\tilde{g} = S^{-1} g$ is the "dual" window of g.

[Wexler, Raz] "Discrete Gabor expansions" (1990)
(Case $n=1$, $C = \gamma \in \mathbb{R}^+$, $B = \beta \in \mathbb{R}^+$)

$$\boxed{\mathcal{G}_{B,C}(g) \text{ is a tight frame} \iff \mathcal{G}_{C^I, B^I}(\tilde{g}) \text{ is an orthogonal system}}$$

Remark: A complete proof of the above result for $C = \gamma > 0$ and $B = \beta > 0$ was provided [Daubechies, H. Landau, Z. Landau] (1995). We shall show the general case.

Two Bessel systems $\{\varphi_j\}, \{\psi_j\} \subset \mathbb{H}$ (Hilbert) are called DUAL if

$$\sum_j{}' \langle f, \varphi_j \rangle \langle \psi_j, g \rangle = \langle f, g \rangle \quad , f, g \in \mathbb{H} \qquad (1)$$

If (1) holds, $(\{\varphi_j\}, \{\psi_j\})$ are reproducing systems i.e

$$\sum_j \langle f, \varphi_j \rangle \psi_j = f \quad \& \quad \sum_j \langle g, \psi_j \rangle \varphi_j = \langle f, g \rangle.$$

Theorem 5.2 has the following generalization:

THEOREM 7.1

Suppose $\{T_{C_p k} g_p : p \in \mathcal{P}, k \in \mathbb{Z}^n\}$ and $\{T_{C_p k} \gamma_p : p \in \mathcal{P}, k \in \mathbb{Z}^n\}$ are Bessel and satisfy L.I.C. These two systems are <u>dual</u> to each other iff

$$\sum_{p \in \mathcal{P}_\alpha} \frac{1}{|\det C_p|} \overline{\hat{g}_p(\xi)}\, \hat{\gamma}_p(\xi + \alpha) = \delta_{\alpha,0} \quad \text{a.e. } \xi \in \mathbb{R}^n$$

$$\text{for all } \alpha \in \Lambda = \bigcup_p C_p^I(\mathbb{Z}^n)$$

THEOREM 7.2 (Wexler-Raz generalized)

$\mathcal{G}_{B,C}(g) = \{T_{Ck} M_{Bm} g : k, m \in \mathbb{Z}^n\}$ and $\mathcal{G}_{B,C}(\gamma) = \{T_{Ck} M_{Bm} \gamma : k, m \in \mathbb{Z}^n\}$ are Bessel systems. These two systems are <u>dual</u> to each other iff

$$\langle g, T_{B^I k} M_{C^I \ell} \gamma \rangle = |\det B||\det C|\, \delta_{k,0} \delta_{\ell,0}, \quad k, \ell \in \mathbb{Z}^n$$

P/ By theorem 7.1 with $\mathcal{P} = \mathbb{Z}^n$, $g_m = M_{Bm} g$, $\gamma_m = M_{Bm}\gamma$ and $C_m = C$ shows that duality of $\mathcal{G}_{B,C}(g)$ and $\mathcal{G}_{B,C}(\gamma)$ is equivalent to

$$F_\ell(\xi) \equiv \sum_{m \in \mathbb{Z}^n} \frac{1}{|\det C|} \hat{g}(\xi - Bm)\, \overline{\hat{g}(\xi - Bm + C^I \ell)} = \delta_{\ell,0} \quad \text{a.e. } \xi \in \mathbb{R}^n \tag{1}$$

for all $\ell \in \mathbb{Z}^n$ (recall that $\Lambda = C^I(\mathbb{Z}^n)$ and $\mathcal{P}_\alpha = \mathbb{Z}^n$ for all $\alpha \in \Lambda$). From (1) we deduce that $F_\ell(\xi) \in L^1(B[0,1]^n)$ and it is $B\mathbb{Z}^n$-periodic. Its Fourier coefficients are

$$\hat{F}_\ell(k) = \frac{1}{|\det B|} \int_{B[0,1]^n} F_\ell(\xi)\, e^{-2\pi i B^I k \cdot \xi}\, d\xi$$

$$= \frac{1}{|\det B||\det C|} \int_{B[0,1]^n} \sum_{m \in \mathbb{Z}^n} M_{-B^I k} \hat{g}(\xi - Bm)\, \overline{T_{-C^I \ell} \hat{g}(\xi - Bm)}\, d\xi$$

$\underset{=}{\text{Periodize}}\ \dfrac{1}{|\det B||\det C|}\displaystyle\int_{\mathbb{R}^n} M_{-B^{I}k}\hat{g}(\xi)\,\overline{T_{-C^{I}\ell}\hat{\gamma}(\xi)}\,d\xi$

$= \dfrac{1}{|\det B||\det C|}\langle M_{-B^{I}k}\hat{g}, T_{-C^{I}\ell}\hat{\gamma}\rangle$

$\underset{=}{\text{Plancherel}}\ \dfrac{1}{|\det B||\det C|}\langle T_{B^{I}k}g, M_{-C^{I}\ell}\gamma\rangle$

$= \dfrac{1}{|\det B||\det C|}\langle g, T_{-B^{I}k}M_{-C^{I}\ell}\gamma\rangle$

Since $F_\ell(\xi) = \delta_{\ell,0}$ a.e. for all $\ell \in \mathbb{Z}^n$, we deduce

$$\langle g, T_{B^{I}k}M_{-C^{I}\ell}\gamma\rangle = |\det B||\det C|\,\delta_{k,0}\,\delta_{\ell,0}$$

as wanted. ■

———— x ————

The systems $\{\varphi_j\}$ and $\{\psi_j\} \subset \mathbb{H}$ are called BIORTHOGONAL to each other if

$$\langle \varphi_k, \psi_\ell\rangle = c\,\delta_{k,\ell}$$

COROLLARY 7.3.

Same conditions as in theorem 7.2. The systems $\mathcal{G}_{B,C}(g)$ and $\mathcal{G}_{B,C}(\gamma)$ are duals to each other $\Leftrightarrow$ $\mathcal{G}_{C^{I},B^{I}}(g)$ and $\mathcal{G}_{C^{I},B^{I}}(\gamma)$ are biorthogonal (with constant $|\det B||\det C|$.

P/ Given $k, k', \ell, \ell' \in \mathbb{Z}^n$ and writing

$$s = e^{2\pi i C^{I}\ell' \cdot B^{I}(k-k')}$$

we deduce

$$< T_{B^I k'} M_{C^I \ell'} g , T_{B^I k} M_{C^I \ell} \gamma > =$$

$$= < g , M_{-C^I \ell'} T_{B^I (k-k')} M_{C^I \ell} \gamma >$$

$$= < g , e^{-2\pi i C^I \ell' \cdot B^I (k-k')} T_{B^I (k-k')} M_{C^I (\ell-\ell')} \gamma >$$

$$= s < g , T_{B^I (k-k')} M_{C^I (\ell-\ell')} \gamma >$$

Assuming $\mathcal{G}_{B,C}(g)$ and $\mathcal{G}_{B,C}(\gamma)$ are duals, theorem 7.2 and the above calculations give the biorthogonality since

$$s < g , T_{B^I(k-k')} M_{C^I(\ell-\ell')} \gamma > = s\, \delta_{k,k'} \delta_{\ell,\ell'} |\det B| |\det C|$$

If $\mathcal{G}_{C^I, B^I}(g)$ and $\mathcal{G}_{C^I, B^I}(\gamma)$ are biorthogonal, in particular $< g, T_{B^I k} M_{C^I \ell} \gamma > = |\det B| |\det C| \delta_{0,k} \delta_{0,\ell}$ and duality follows from theorem 7.2. ■

COROLLARY 7.4 (Wexler-Raz)

The Gabor system is a PF iff $\mathcal{G}_{C^I, B^I}(g)$ is an orthogonal system (with constant $|\det B| |\det C|$).

P/ If $\mathcal{G}_{B,C}(g)$ is a PF, $\tilde{g} = S^{-1} g = g$. Apply Cor 7.3. ■

From Wexler-Raz (original version stated in 1990) we can prove Rieffel's density theorem ("Von Neumann algebras associated to a pair of lattices on Lie groups" 1981)

COROLLARY 7.5 (Generalized Rieffel's)

Let $B, C \in Gl_n(\mathbb{R})$ and $g \in L^2(\mathbb{R}^n)$. If $|\det B||\det C| > 1$, the set $\mathcal{G}_{B,C}(g) = \{T_{Ck}M_{Bm}g : k, m \in \mathbb{Z}^n\}$ <u>cannot</u> be a frame.

P/ If $\mathcal{G}_{B,C}(g)$ were a frame, for $f \in L^2(\mathbb{R}^n)$

$$f = \sum_{k,m} \langle f, T_{Ck}M_{Bm}\tilde{g}\rangle T_{Ck}M_{Bm}g$$

(conv. in $L^2(\mathbb{R}^d)$) where $\tilde{g} = S^{-1}g$ is the "dual window".

By the minimality property of the dual frame (prop 2.4) if we can write

$$f = \sum_{k,m} \alpha_{k,m} T_{Ck}M_{Bm}g \quad , \quad \alpha_{k,m} \in \ell^2$$

we must have

$$\sum_{k,m} |\langle f, T_{Ck}M_{Bm}\tilde{g}\rangle|^2 \leq \sum_{k,m} |\alpha_{k,m}|^2 \qquad (1)$$

Since

$$g = \sum_{k,m} \delta_{k,0}\delta_{m,0} T_{Ck}M_{Bm}g$$

from (1) we deduce

$$|\langle g, \tilde{g}\rangle|^2 \leq \sum_{k,m} |\langle g, T_{Ck}M_{Bm}\tilde{g}\rangle|^2 \leq \sum_{k,m} |\delta_{k,0}\delta_{m,0}|^2 = 1$$

By Wexler-Raz (Thm 7.2) with $\tilde{g} = \gamma$ we have $\langle g, \tilde{g}\rangle = |\det B||\det C|$ $\therefore$ $|\det B||\det C| \leq 1$ ∎

For a given $g \in L^2(\mathbb{R}^n)$, if $B = \beta I_n$ and $C = \gamma I_n$ with $\beta, \gamma > 0$, by Rieffel's result

$$\mathcal{G}_{\beta,\gamma}(g) = \{T_{\gamma k} M_{Bm} g : k, m \in \mathbb{Z}^n\}$$

is not a frame if $\beta\gamma > 1$. But $\mathcal{G}_{\beta,\gamma}(g)$ may not be a frame even if $\beta\gamma \leq 1$

<u>QUESTION</u>. For $g \in L^2(\mathbb{R}^n)$, find the range of values of $\beta\gamma$ (that is a subset $E_g \subset (0,1]$) such that

$$\mathcal{G}_{\beta,\gamma}(g) \text{ frame } \Leftrightarrow \beta\gamma \in E_g$$

<u>REMARK</u>. For $g(x) = c e^{-\pi x^2}$ $(n=1)$, [Lyubarski] and [Seip, Wallstein] showed in (1992) that $E_g = (0,1)$

8. CHARACTERIZATION OF WAVELET TYPE SYSTEMS THAT ARE PARSEVAL FRAMES

We consider now systems of the form

$$W_A(\psi) = \{ D_{A^j} T_k \psi : j \in \mathbb{Z}, k \in \mathbb{Z}^n \} \qquad (1)$$

where $A \in Gl_n(\mathbb{R})$ and $\psi \in L^2(\mathbb{R}^n)$. To be able to apply theorem 5.2 we need to write (1) in de form

$$\{ T_{C_p k} g_p : p \in \mathcal{P}, k \in \mathbb{Z}^n \} \qquad (2)$$

Notice that

$$D_{A^j} T_k \psi(x) = |\det A|^{j/2} \psi(A^j x - k) = |\det A|^{j/2} \psi(A^j(x - A^{-j}k))$$
$$= T_{A^{-j}k} D_{A^j} \psi(x)$$

Thus, we take $\mathcal{P} = \mathbb{Z}$, $g_j = D_{A^j}\psi$ and $C_j = A^{-j}$ ($j \in \mathbb{Z}$).

REMARK 1. We must consider L.I.C for (1). This is a delicate matter. It can be shown that if A is <u>expansive</u> (i.e. all eigenvalues have modulus >1), then $W_A(\psi)$ satisfies L.I.C. Also, there are matrices which satisfy L.I.C but are not expansive, for example

$$A = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}, \quad a > 1.$$

REMARK 2. Characterize all matrices for which $W_A(\psi)$ satisfies L.I.C is an open problem. Another open problem is to characterize all matrices A for which $\exists\, \psi \in L^2(\mathbb{R}^n)$ such that $W_A(\psi)$ is a PF (or a frame).

(Assume L.I.C)

By theorem 5.2, $W_A(\psi)$ is a PF $\Leftrightarrow$

$$\sum_{j\in P_\alpha} \frac{1}{|\det A^j|} (D_A^j\psi)^\wedge(\xi)\overline{(D_A^j\psi)^\wedge(\xi+\alpha)} = \delta_{\alpha,0} \quad \text{a.e. } \xi\in\mathbb{R}^n \qquad (3)$$

for all $\alpha\in\Lambda = \bigcup_{j\in\mathbb{Z}} (A^t)^j \mathbb{Z}^n$ and $P_\alpha = \{j\in\mathbb{Z} : (A^t)^{-j}\alpha\in\mathbb{Z}^n\}$

Since $(D_A^j\psi)^\wedge(\xi) = D_{(A^t)^{-j}}\hat\psi(\xi)$, eq (3) becomes

$$\sum_{j\in P_\alpha}{}' \hat\psi((A^t)^{-j}\xi)\overline{\hat\psi((A^t)^{-j}(\xi+\alpha))} = \delta_{\alpha,0} \quad \text{a.e. } \xi\in\mathbb{R}^n. \qquad (4)$$

for all $\alpha\in\Lambda$.

The case $\alpha=0$ of Eq (4) is

$$\sum_{j\in\mathbb{Z}}{}' |\hat\psi((A^t)^{-j}\xi)|^2 = 1 \quad \text{a.e. } \xi\in\mathbb{R}^n \qquad (5)$$

which is known as Calderón's condition

There are many equations in (4): one for each $\alpha\in\Lambda$. But many of them are repeated. Indeed, it is enough to have one equation for each $m\in\mathbb{Z}^n$.

LEMMA 8.1.

Eq (4) is valid for all $\alpha\in\Lambda$ $\Leftrightarrow$

$$\sum_{j\in P_m}{}' \hat\psi((A^t)^{-j}\xi)\overline{\hat\psi((A^t)^{-j}(\xi+m))} = \delta_{m,0} \quad \text{a.e. } \xi\in\mathbb{R}^n \qquad (6)$$

for all $m\in\mathbb{Z}^n$, where $P_m = \{j\in\mathbb{Z} : (A^t)^{-j}m\in\mathbb{Z}^n\}$.

P/ Enough to prove that (6) $\Rightarrow$ (4). The case $\alpha=0$ in (4) corresponds to the case $m=0$ in (6). Let $\alpha\in\Lambda\setminus\{0\}$. Then

$$\alpha = (A^t)^{+j_0} m_0 \quad \text{for some } j_0\in\mathbb{Z},\ m_0\in\mathbb{Z}^n\setminus\{0\}.$$

Then

$$\sum_{j\in P_\alpha}{}' \hat{\varphi}((A^t)^{-j}\xi)\,\overline{\hat{\varphi}((A^t)^{-j}(\xi+\alpha))} \;\overset{\xi=(A^t)^{j_0}\eta}{=\!=\!=}$$

$$= \sum_{j\in P_{(A^t)^{j_0}m_0}} \hat{\varphi}((A^t)^{-j+j_0}\eta)\,\overline{\hat{\varphi}((A^t)^{-j+j_0}(\eta+m_0))} \qquad (7)$$

With $k=j-j_0$, $P_\alpha = P_{(A^t)^{j_0}m_0} = \{j\in\mathbb{Z} : (A^t)^{-j}(A^t)^{j_0}m_0\in\mathbb{Z}^n\}$,
so that $j=k+j_0\in P_{(A^t)^{j_0}m_0} \Leftrightarrow k\in P_{m_0}=\{k\in\mathbb{Z}^n : (A^t)^{-k}m_0\in\mathbb{Z}^n\}$.
Thus, the left hand side of (7) coincides with

$$\sum_{k\in P_{m_0}}{}' \hat{\varphi}((A^t)^{-k}\eta)\,\overline{\hat{\varphi}((A^t)^{-k}(\eta+m_0))}$$

which is 0 by (6), since $m_0\neq 0$. ■

For special cases of matrices A equations (6) becomes simpler. Let us consider that A has integer entries (i.e. $A(\mathbb{Z}^n)\subset\mathbb{Z}^n$) and that A is <u>expanding</u>.

SOME RESULTS ABOUT EXPANDING MATRICES

A matrix $M\in GL_n(\mathbb{R})$ is <u>expanding</u> if $|\lambda|>1$ for all $\lambda\in\sigma(M)$. It can be shown that M is expanding iff there exist $0<k\le 1<\gamma<\infty$ such that

$$\|M^j x\| \ge k\gamma^j\|x\| \qquad (8)$$

for all $x\in\mathbb{R}^n$, $j\in\mathbb{Z}$, $j\ge 0$. From (8), applied to $y=M^{-j}x$ it can be deduced

$$\|M^{-j}x\| \le \frac{1}{k}\gamma^{-j}\|x\| \qquad (9)$$

for all $x\in\mathbb{R}^n$, $j\in\mathbb{Z}$, $j\ge 0$.

LEMMA 8.2

Let $M \in Gl_n(\mathbb{Z})$ expanding. If $m \in \mathbb{Z}^n \setminus \{0\}$, there exists unique $d \in \mathbb{Z}^+ \cup \{0\}$ and $r \in \mathbb{Z}^n \setminus M(\mathbb{Z}^n)$ such that $m = M^d r$.

REMARK: $M = 2$; $m \in \mathbb{Z} \setminus \{0\}$; $m = 2^d r$ with $r \in \mathbb{Z} \setminus 2\mathbb{Z}$

P/
- If $m \notin M(\mathbb{Z}^n)$, then $m = M^0 m$ ✓
- If $m \in M(\mathbb{Z}^n)$, $m = M m_1$ with $m_1 \in \mathbb{Z}^n$. If $m_1 \notin M(\mathbb{Z}^n)$ write $m = M^1 m_1$ ✓
- If $m_1 \in M(\mathbb{Z}^n)$ write $m_1 = M m_2$ with $m_2 \in \mathbb{Z}^n$. If $m_2 \notin M(\mathbb{Z}^n)$, then $m = M^2 m_2$ ✓

Continue in this way. This process must stop. Otherwise

$$m = M^j m_j \quad \text{for all } j \geq 0 \text{ and } m_j \in \mathbb{Z}^n$$

By (a), $\|m_j\| = \|M^{-j} m\| \leq \frac{1}{k} \gamma^{-j} \|m\| \xrightarrow{j \to \infty} 0 \quad (\gamma > 1)$ which will imply $m_j = 0$ for j large, and $m = 0$!! ∎

We can now proved a result "similar" to Gripenberg-Wang characterization.

PROPOSITION 8.3.

Let $\psi \in L^2(\mathbb{R}^n)$ and $A \in Gl_n(\mathbb{Z})$ expanding. Then, $W_A(\psi)$ is a P.F. $\Longleftrightarrow$

$$\sum_{j \in \mathbb{Z}} |\hat{\psi}((A^t)^{-j}\xi)|^2 = 1 \quad \text{a.e. } \xi \in \mathbb{R}^n \qquad (10)$$

and

$$\sum_{j \geq 0} \hat{\psi}((A^t)^j \xi)\, \overline{\hat{\psi}((A^t)^j(\xi + r))} = 0 \quad \text{a.e. } \xi \in \mathbb{R}^n \qquad (11)$$

for all $r \in \mathbb{Z}^n \setminus A^t \mathbb{Z}^n$

P/ Eq (10) is the case $\alpha=0$ of lemma 8.1.
Let $m \in \mathbb{Z}^n - \{0\}$. We need to examine

$$P_m = \{j \in \mathbb{Z} : (A^t)^{-j} m \in \mathbb{Z}^n\}.$$

If $j \in P_m$ we have $(A^t)^{-j} m = s \in \mathbb{Z}^n$. By lemma 8.2 there exist unique $d \in \mathbb{Z}^+ - \{0\}$ and $r \in \mathbb{Z}^n - A^t \mathbb{Z}^n$ s.t. $m = (A^t)^d r$. Hence

$$s = (A^t)^{-j+d} r.$$

We must have $-j+d \geq 0$; otherwise with $-j+d = -\ell < 0$ we deduce $s = (A^t)^{-\ell} r \Rightarrow r = (A^t)^{\ell} s \in (A^t)(\mathbb{Z}^n)$.

Thus, for $m = (A^t)^d r \in \mathbb{Z}^n - \{0\}$, (6) of lemma 8.1 becomes

$$\sum_{j \leq d} \hat{\psi}((A^t)^{-j}\xi) \overline{\hat{\psi}((A^t)^{-j}(\xi + (A^t)^d r))} = 0 \quad \text{a.e. } \xi \in \mathbb{R}^n,$$

with $r \in \mathbb{Z}^n - A^t \mathbb{Z}^n$. Replace ξ by $(A^t)^d \eta$ in the above expression and change index of summation to $k = d - j \geq 0$ to obtain eq (11) of proposition 8.3. ∎

Eq (6) of lemma 8.1 can be used to obtain examples of PF's or even wavelets. The simplest way to satisfy (6) for $m \neq 0$ is to choose $\psi \in L^2(\mathbb{R}^d)$ such that

$$(\operatorname{supp} \hat{\psi}) \cap (\operatorname{supp} \hat{\psi}(\cdot - m)) = \emptyset \quad (\text{a.e.})$$

for all $m \in \mathbb{Z}^n$, $m \neq 0$ (observe that $(A^t)^{-j} m \in \mathbb{Z}^n$ for $j \in P_m$ as in (6)). In this situation, all that is needed to check is Calderón's condition

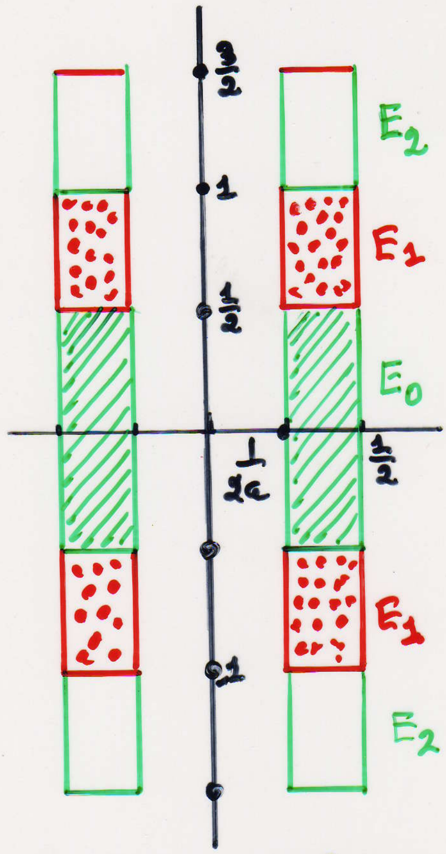
$$\sum_{j\in\mathbb{Z}} |\hat{\psi}((A^t)^{-j}\xi)|^2 = 1 \quad \text{a.e. } \xi \in \mathbb{R}^n \qquad (12)$$

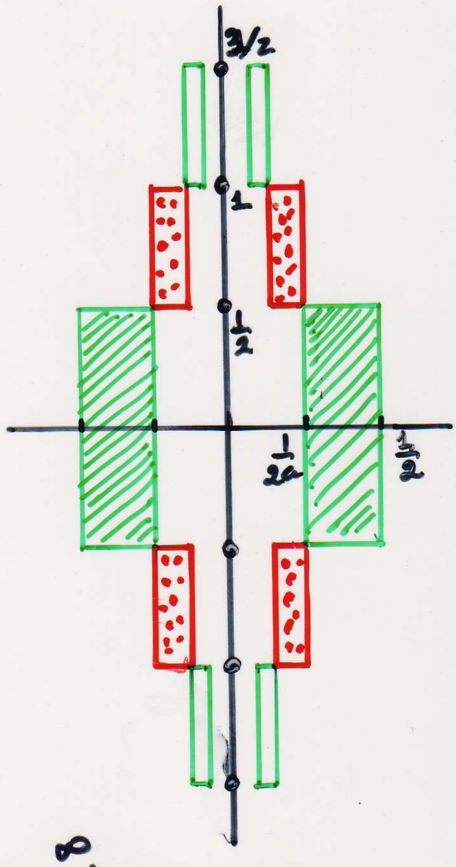
EXAMPLE:

For $A = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$, $a > 1$ (not expansive), let

$$V = \{(x,y) \in \mathbb{R}^2 : \tfrac{1}{2a} \leq \|x\| < \tfrac{1}{2}\}$$

$$E_n = \{(x,y) \in \mathbb{R}^2 : \tfrac{1}{2a} \leq \|x\| < \tfrac{1}{2},\ \tfrac{n}{2} \leq \|y\| < \tfrac{n+1}{2}\},\ n = 0,1,2,\ldots$$





Define $S_n = A^{-n}E_n$, $n = 0,1,2,\ldots$ & $W = \bigcup_{n=0}^{\infty} S_n$. Let

$$\hat{\psi} = \chi_W .$$

Since $\bigcup_{n=0}^{\infty} A^n S_n = \bigcup_{n=0}^{\infty} E_n = V$ and $\{A^j V : j \in \mathbb{Z}^n\}$ is a partition of $\mathbb{R}^2$ we conclude (12).

Moreover,

$$|W| = \sum_{n=0}^{\infty} 4\,\tfrac{1}{2}\left[\frac{1}{2a^n} - \frac{1}{2a^{n+1}}\right] = \sum_{n=0}^{\infty} \frac{1}{a^n} - \sum_{n=0}^{\infty} \frac{1}{a^{n+1}} = 1,$$

so that $W_A(\psi)$ is an orthonormal wavelet.