4. GABOR AND WAVELET FRAMES

4.1. GABOR FRAMES : $g \in L^2(\mathbb{R}^n)$, $B, C \in GL_n(\mathbb{R})$

$$\mathcal{G} = \mathcal{G}_{B,C}(g) = \{ T_{Ck} M_{Bm} g : k, m \in \mathbb{Z}^n \}$$

THEOREM 4.1. Let $\mathcal{G} = \mathcal{G}_{B,C}(g)$ be a frame in $L^2(\mathbb{R}^n)$. Let $\tilde{g} = S^{-1} g$ where $S = A^* A$ is the frame operator for $\mathcal{G}$. Then, the dual frame $\tilde{\mathcal{G}} = \{ S^{-1}(T_{Ck} M_{Bm} g) : k, m \in \mathbb{Z}^n \}$ satisfies

$$S^{-1}(T_{Ck} M_{Bm} g) = T_{Ck} M_{Bm} S^{-1} g \quad , k, m \in \mathbb{Z}^n.$$

(i.e. $\tilde{\mathcal{G}} = \mathcal{G}_{B,C}(\tilde{g})$).

P/ Enough to prove that S commutes with $T_{Ck} M_{Bm}$.
For $f \in L^2(\mathbb{R}^n)$ ~~and~~ ~~[illegible]~~

$$(T_{Ck} M_{Bm})^{-1} S \, T_{Ck} M_{Bm} f =$$

$$\sum_{r,s} \langle T_{Ck} M_{Bm} f , T_{Cr} M_{Bs} g \rangle (T_{Ck} M_{Bm})^{-1} T_{Cr} M_{Bs} g \qquad (1)$$

But

$$(T_{Ck} M_{Bm})^{-1} T_{Cr} M_{Bs} g(x) = M_{-Bm} T_{-Ck} T_{Cr} M_{Bs} g(x)$$

$$= e^{-2\pi i Bm \cdot C(r-k)} T_{C(r-k)} M_{B(s-m)} g(x) \qquad (2)$$

From (1) and (2)

$$(T_{Ck} M_{Bm})^{-1} S (T_{Ck} M_{Bm}) \cancel{f(x)} =$$

$$= \sum_{r,s}{}' \langle f , M_{-Bm} T_{-Ck} T_{Cr} M_{Bs} g \rangle M_{-Bm} T_{-Ck} T_{Cr} M_{Bs} g \, (x)$$

$$= \sum_{r,s} \langle f, T_{C(r-k)} M_{B(s-m)} g \rangle \, T_{C(r-k)} M_{B(s-m)} g = Sf$$

(because the exp. factor cancels) ~~and~~ performing a change of indices. ■

4.2. WAVELET FRAMES : $\psi \in L^2(\mathbb{R}^n)$, $A \in GL_n(\mathbb{R})$

$$\mathcal{W} = \mathcal{W}_A(\psi) = \{ D_{A^j} T_{Ck} \psi : k \in \mathbb{Z}^n, \; j \in \mathbb{Z} \}$$

The frame theory produces a dual frame form by the elements

$$\widetilde{D_{A^j} T_{Ck} \psi} = S^{-1}(D_{A^j} T_{Ck} \psi)$$

where S is the frame operator for $\mathcal{W}$.

LEMMA 4.2

The frame operator S, and hence S^{-1}, commutes with dilations D_{A^ℓ}, $\ell \in \mathbb{Z}$.

P/ For $f \in L^2(\mathbb{R}^n)$

$$(S D_{A^\ell} f)(x) = \sum_{j,k} \langle D_{A^\ell} f, D_{A^j} T_{Ck} \psi \rangle \, D_{A^j} T_{Ck} \psi(x)$$

$$= \sum_{j,k} \langle f, D_{A^{j-\ell}} T_{Ck} \psi \rangle \, D_{A^j} T_{Ck} \psi(x)$$

$$= D_{A^\ell} \Big(\sum_{j,k} \langle f, D_{A^{j-\ell}} T_{Ck} \psi \rangle \, D_{A^{j-\ell}} T_{Ck} \psi(x) \Big)$$

$$= D_{A^\ell} (Sf)(x)$$ ■

But, S does <u>not</u> commutes with translations:

$$S(T_{cm}f)(x) = \sum_{j,k} \langle T_{cm} f, D_{A^j} T_{ck}\psi \rangle D_{A^j} T_{ck}\psi(x)$$

$$= \sum_{j,k} \langle f, T_{-cm} D_{A^j} T_{ck}\psi \rangle D_{A^j} T_{ck}\psi(x)$$

$$= \sum_{j,k} \langle f, D_{A^j} T_{-A^j cm} T_{ck}\psi \rangle D_{A^j} T_{ck}\psi(x)$$

and the problem is that not allways $A^j cm \in \mathbb{Z}^n$.

REMARK. The fact that S does not necessarily commutes with translations does not proved that the dual frame $\tilde{W}$ cannot be generated by a single function $\tilde{\psi}$.

But in [Hernández, Weiss] "A first course on wavelets" (1996) (§ 8.3) a frame for $L^2(\mathbb{R})$, $A=2$, $c=1$, is given and proved (by contradiction) that its dual frame cannot be generated by a single function $\tilde{\psi}$.

Two Characterizations

GRIPENBERG (1995), X. WANG (1995)

$\{D_{2^j}T_k\psi : k \in \mathbb{Z}, j \in \mathbb{Z}\}$ is a PF $\Longleftrightarrow$

(1) $\sum_{j\in\mathbb{Z}} |\hat{\psi}(2^j\xi)|^2 = 1$ a.e. $\xi \in \mathbb{R}$

(2) $t_q(\xi) = \sum_{j\geq 0} \hat{\psi}(2^j\xi)\overline{\hat{\psi}(2^j(\xi+q))} = 0$ a.e. $\xi \in \mathbb{R}$ for all $q \in 2\mathbb{Z}+1$

RON-SHEN (1997), CASAZZA-CHRISTENSEN-JANSSEN (1999), W. CZAJA (2000)

$\{T_{\gamma k} M_{\beta m} g : k, m \in \mathbb{Z}^n\}$ is a PF $\Longleftrightarrow$

$\sum'_{m\in\mathbb{Z}^n} \hat{g}(\xi-\beta m)\overline{\hat{g}(\xi-\beta m+\frac{1}{\gamma}u)} = \gamma\delta_{u,0}$ a.e. $\xi \in \mathbb{R}^n$ for all $u \in \mathbb{Z}^n$

5. A UNIFIED THEORY FOR PARSEVAL FRAMES

Let $\mathcal{P}$ be a collection of indices, $\{g_p : p \in \mathcal{P}\} \subset L^2(\mathbb{R}^n)$ and $\{C_p : p \in \mathcal{P}\} \subset GL_n(\mathbb{R})$. Consider

$$\{T_{C_p k}\, g_p : k \in \mathbb{Z}^n,\ p \in \mathcal{P}\} \qquad (1)$$

EXAMPLE 1. Gabor systems

$$\mathcal{G} = \mathcal{G}_{B,C}(\{g^1, \dots, g^L\}) = \{T_{Ck} M_{Bm} g^\ell : k, m \in \mathbb{Z}^n,\ \ell = 1, 2, \dots, L\}$$

are of the form (1) taking $\mathcal{P} = \mathbb{Z}^n \times \{1, 2, \dots, L\}$, $C_p = C_{(m,\ell)} = C$ for all $p \in \mathcal{P}$ and $g_p = g_{(m,\ell)} = M_{Bm} g^\ell$ for all $p \in \mathcal{P}$.

EXAMPLE 2. Wavelet systems

$$\mathcal{W} = \mathcal{W}_{A,C}(\{\psi^1, \dots, \psi^L\}) = \{D_{A^j} T_{Ck} \psi^\ell : j \in \mathbb{Z}, k \in \mathbb{Z}^n; \ell = 1, 2, \dots, L\}$$

are of the form (1). Observe that

$$D_{A^j} T_{Ck} \psi^\ell = T_{A^{-j} Ck} D_{A^j} \psi^\ell .$$

Take $\mathcal{P} = \mathbb{Z} \times \{1, 2, \dots, L\}$, $C_p = C_{(j,\ell)} = A^{-j} C$ and $g_p = D_{A^j} \psi^\ell$

<u>NOTATION</u>.

$$\Lambda = \bigcup_{p \in \mathcal{P}} C_p^{I}(\mathbb{Z}^n) \qquad \text{where} \quad C_p^{I} = (C_p^{t})^{-1}$$

If $\alpha \in \Lambda$ we write $\mathcal{P}_\alpha = \{p \in \mathcal{P} : C_p^t \alpha \in \mathbb{Z}^n\}$

Example 1. For $\mathcal{G}_{B,C}(g)$, $\mathcal{P} = \mathbb{Z}^n$ and $\Lambda = C^I(\mathbb{Z}^n)$.
If $\alpha \in \Lambda$, $\alpha = C^I m$ for some $m \in \mathbb{Z}^n$. Thus $\mathcal{P}_\alpha = \mathbb{Z}^n$ for all $\alpha \in \Lambda$.

Example 2. For $W_{2,1}(\psi)$ we have $\mathcal{P} = \mathbb{Z}$ and $\Lambda = \bigcup_{p\in\mathbb{Z}} 2^p\mathbb{Z}$. For example, if $\alpha = 2^{-3} \in \Lambda$

$$\mathcal{P}_{2^{-3}} = \{p\in\mathbb{Z} : 2^{-p}\cdot 2^{-3} \in \mathbb{Z}\} = \{-3,-4,-5,\dots\}$$

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To prove that $\{\varphi_j\}\subset\mathbb{H}$ is a PF it is enough to consider a dense subset $\mathcal{D}\subset\mathbb{H}$.

LEMMA 5.1. (Chapter 7 of [HW])

$\{\varphi_j\}\subset\mathbb{H}$ is a PF $\Leftrightarrow$ $\sum |\langle f,\varphi_j\rangle|^2 = \|f\|^2$ for all $f\in\mathcal{D}$, dense subset of $\mathbb{H}$.

P/ $\Leftarrow$) If $f\in\mathbb{H}$ let $\{f_n\}\subset\mathcal{D}$ s.t. $\lim_{n\to\infty}\|f_n-f\|_{\mathbb{H}}=0$. Then

$$\sum_{j=1}^{N} |\langle f,\varphi_j\rangle|^2 = \lim_{n\to\infty}\sum_{j=1}^{N}|\langle f_n,\varphi_j\rangle|^2 \le \lim_{n\to\infty}\sum_{j=1}^{\infty}|\langle f_n,\varphi_j\rangle|^2$$

$$= \lim_{n\to\infty}\|f_n\|^2 = \|f\|^2$$

Since N is arbitrary we deduce $\displaystyle\sum_{j=1}^{\infty}|\langle f,\varphi_j\rangle|^2 \le \|f\|^2$. (1)

Now choose $\varepsilon>0$ and $g\in\mathcal{D}$ s.t. $\|f-g\|<\varepsilon$. Thus, $\|f\|\le\|g\|+\varepsilon$ and

$$\|f\|-2\varepsilon \le \|g\|-\varepsilon \le \|g\|-\|g-f\| = \Big(\sum_{j=1}^{\infty}|\langle g,\varphi_j\rangle|^2\Big)^{1/2} - \|g-f\|$$

$$\overset{(1)}{\le} \Big(\sum_{j=1}^{\infty}|\langle g,\varphi_j\rangle|^2\Big)^{1/2} - \Big(\sum_{j=1}^{\infty}|\langle g-f,\varphi_j\rangle|^2\Big)^{1/2} \overset{T.I}{\le} \Big(\sum_{j=1}^{\infty}|\langle f,\varphi_j\rangle|^2\Big)^{1/2}$$

The result follows since $\varepsilon>0$ is arbitrary. ∎

In our proof we shall take

$$\mathcal{D} = \{ f \in L^2(\mathbb{R}^n) : \hat{f} \in L^\infty(\mathbb{R}^n) \text{ and } \operatorname{sop} \hat{f} \text{ compact} \}$$

which is dense in $L^2(\mathbb{R}^n)$.

Our main result about PF's of the type (1) requires a technical hypothesis called LOCAL INTEGRABILITY CONDITION (L.I.C): for all $f \in \mathcal{D}$

$$L(f) = \sum_{p\in\mathcal{P}} \sum_{m\in\mathbb{Z}^n} \int_{\operatorname{sop}\hat{f}} |\hat{f}(\xi + C_p^I m)|^2 \frac{1}{|\det C_p|} |\hat{g}_p(\xi)|^2 d\xi < \infty$$

THEOREM 5.2.

Suppose $\{T_{C_p k} g_p : p \in \mathcal{P}, k \in \mathbb{Z}^n\}^{(3)}$ satisfies L.I.C. Then, (3) is a PF $\Leftrightarrow$

$$\sum_{p\in\mathcal{P}_\alpha} \frac{1}{|\det C_p|} \overline{\hat{g}_p(\xi)}\, \hat{g}_p(\xi+\alpha) = \delta_{\alpha,0} \quad \text{a.e. } \xi\in\mathbb{R}^n \quad \text{for all } \alpha \in \Lambda \qquad (4)$$

REMARK. For $\alpha = 0$, $\mathcal{P}_0 = \mathcal{P}$ and equation (4) becomes

$$\sum_{p\in\mathcal{P}} \frac{1}{|\det C_p|} |\hat{g}_p(\xi)|^2 = 1 \quad \text{a.e. } \xi \in \mathbb{R}^n .$$

Hence, by using C-S, all the series in (4) converge absolutely.

Theorem 5.2 will be proved as a consequence of several lemmas.

For $f, g \in L^2(\mathbb{R}^n)$ and $C \in M_{n\times n}(\mathbb{R})$

$$[f,g]_C(x) = \sum_{k\in\mathbb{Z}} f(x-Ck)\overline{g(x-Ck)}$$

is a $C\mathbb{Z}^n$-periodic function.

LEMMA 5.3.

Let $C \in GL_n(\mathbb{R})$ and $C^I = (C^t)^{-1}$. If $f \in \mathcal{S}$ and $g \in L^2(\mathbb{R}^n)$,

$$\sum_{k\in\mathbb{Z}^n} |\langle f, T_{Ck}g\rangle|^2 = \frac{1}{|\det C|}\int_{C^I([0,1)^n)} |[\hat{f},\hat{g}]_{C^I}(\xi)|^2 d\xi$$

P/

$$\sum_{k\in\mathbb{Z}^n} |\langle f, T_{Ck}g\rangle|^2 \overset{\text{Planch}}{=} \sum_{k\in\mathbb{Z}^n} \left|\int_{\mathbb{R}^n} \hat{f}(\xi)\overline{\hat{g}(\xi)}\, e^{2\pi i Ck\cdot\xi} d\xi\right|^2$$

$\mathbb{R}^n = \biguplus_{\ell\in\mathbb{Z}^n} C^I(\mathbb{T}^n - \ell)$

$$= \sum_{k\in\mathbb{Z}^n} \left|\sum_{\ell\in\mathbb{Z}^n} \int_{C^I(\mathbb{T}^n)} \hat{f}(\xi - C^I\ell)\overline{\hat{g}(\xi - C^I\ell)}\, e^{2\pi i Ck\cdot\xi} d\xi\right|^2$$

$$= \sum_{k\in\mathbb{Z}^n} \Bigg|\underbrace{\int_{C^I(\mathbb{T}^n)} [\hat{f},\hat{g}]_{C^I}(\xi)\, e^{2\pi i Ck\cdot\xi} d\xi}_{\approx \text{ Fourier coefficients of } [\hat{f},\hat{g}]_{C^I},\ C^I\mathbb{Z}^n\text{-per}}\Bigg|^2$$

$\{\sqrt{|\det C|}\, e^{2\pi i Ck\cdot\xi}\}_{k\in\mathbb{Z}^n}$ is an o.n. basis of $C^I(\mathbb{T}^n)$

$$\overset{\text{Planch.}}{=} \frac{1}{|\det C|}\int_{C^I(\mathbb{T}^n)} |[\hat{f},\hat{g}]_{C^I}(\xi)|^2 d\xi$$

∎

LEMMA 5.4.

Let $C\in Gl_n(\mathbb{R})$, $C^I=(C^t)^{-1}$, $f\in\mathcal{S}$ and $g\in L^2(\mathbb{R}^n)$. The function $H(x)=\sum_{k\in\mathbb{Z}^n}|\langle T_x f, T_{Ck} g\rangle|^2$ is a trigonometric pol. of the form

$$H(x)=\sum_{m\in\mathbb{Z}^n}\hat{H}(m)\, e^{2\pi i (C^I m)\cdot x}, \quad x\in\mathbb{R}^n$$

where

$$\hat{H}(m)=\frac{1}{|\det C|}\int_{\mathbb{R}^n}\hat{f}(\xi)\,\overline{\hat{f}(\xi+C^I m)}\;\overline{\hat{g}(\xi)}\;\hat{g}(\xi+C^I m)\,d\xi \qquad (5)$$

and only a finite number of these expressions are non-zero.

P/ The last assertion follows from $f\in\mathcal{S}$.

$$|\det C|\,H(x) \overset{\text{Lemma 5.3}}{=\!=\!=} \int_{C^I(\mathbb{T}^n)} \left|\left[(T_x f)^\wedge, \hat{g}\right]_{C^I}(\xi)\right|^2 d\xi$$

$$=\int_{C^I(\mathbb{T}^n)}\Big|\sum_{m\in\mathbb{Z}^n} e^{-2\pi i(\xi+C^I m)\cdot x}\,\hat{f}(\xi+C^I m)\,\overline{\hat{g}(\xi+C^I m)}\Big|^2 d\xi$$

$|e^{-2\pi i\xi\cdot x}|=1$

$$=\int_{C^I(\mathbb{T}^n)}\Big(\sum_{m\in\mathbb{Z}^n} e^{-2\pi i C^I m\cdot x}\,\hat{f}(\xi+C^I m)\,\overline{\hat{g}(\xi+C^I m)}\Big)$$

$$\times\Big(\sum_{\ell\in\mathbb{Z}^n} e^{2\pi i C^I \ell\cdot x}\,\overline{\hat{f}(\xi+C^I \ell)}\,\hat{g}(\xi+C^I \ell)\Big)d\xi$$

Write $k=\ell-m$, multiply and periodize in m

$$=\sum_{k\in\mathbb{Z}^n}\Big(\int_{\mathbb{R}^n}\hat{f}(\xi)\,\overline{\hat{f}(\xi+C^I k)}\;\hat{g}(\xi)\,\overline{\hat{g}(\xi+C^I k)}\,d\xi\Big)e^{2\pi i C^I k\cdot x}$$

Interchanges are possible because $f\in\mathcal{S}$. ∎

LEMMA 5.5.

Suppose $\{T_{C_p k} g_p : k \in \mathbb{Z}^n, p \in \mathcal{P}\}$ satisfies L.I.C. If $f \in \mathcal{D}$, the function

$$\omega_f(x) \equiv N^2(T_x f) = \sum_{p\in\mathcal{P}} \sum_{k\in\mathbb{Z}^n} |\langle T_x f, T_{C_p k} g_p\rangle|^2, \quad x \in \mathbb{R}^n$$

is a continuous function that coincides pointwise with its absolutely convergent (almost periodic) Fourier series

$$\sum_{\alpha\in\Lambda}' \hat{\omega}_f(\alpha)\, e^{2\pi i \alpha\cdot x}$$

where

$$\hat{\omega}_f(\alpha) = \int_{\mathbb{R}^n} \hat{f}(\xi)\, \overline{\hat{f}(\xi+\alpha)} \sum_{p\in\mathcal{P}_\alpha}' \frac{1}{|\det C_p|}\, \overline{\hat{g}_p(\xi)}\, g_p(\xi+\alpha)\, d\xi$$

and this last integral converges absolutely.

P/ $\displaystyle \omega_f(x) \equiv N^2(T_x f) = \sum_{p\in\mathcal{P}} \underbrace{\sum_{k\in\mathbb{Z}^n} |\langle T_x f, T_{C_p k} g_p\rangle|^2}_{\omega_{f,p}(x)}$

By lemma 5.4, $\omega_{f,p}(x)$ is a $C_p\mathbb{Z}^n$-per. trig. poly. of the form

$$\omega_{f,p}(x) = \sum_{m\in\mathbb{Z}^n} \hat{\omega}_{p,f}(m)\, e^{2\pi i C_p^I m\cdot x} \qquad (6)$$

where $\hat{\omega}_{p,f}(m)$ is given by (5) replacing C by C_p.
The L.I.C condition is used to prove that $\{\hat{\omega}_{f,p}(m) : p\in\mathcal{P}, m\in\mathbb{Z}^n\} \in \ell^1(\mathcal{P}\times\mathbb{Z}^n)$ so that convergence in (6) is absolute and uniform.

With $\Lambda = \bigcup_{p\in\mathcal{P}} C_p^I(\mathbb{Z}^n)$ and $\mathcal{P}_\alpha = \{p\in\mathcal{P} : C_p^t\alpha \in \mathbb{Z}^n\}$ we can write

$$\omega_f(x) = \sum_{\alpha\in\Lambda}' \left\{ \sum_{p\in\mathcal{P}_\alpha} \frac{1}{|\det C_p|} \int_{\mathbb{R}^n} \hat{f}(\xi)\, \overline{\hat{f}(\xi+\alpha)}\, \overline{\hat{g}_p(\xi)}\, \hat{g}_p(\xi+\alpha)\, d\xi \right\} e^{2\pi i\alpha\cdot x}$$

$$= \sum_{\alpha\in\Lambda} \hat{\omega}(\alpha)\, e^{2\pi i \alpha\cdot x}. \qquad \blacksquare$$

LEMMA 5.6.

Let $\{c_\alpha : \alpha \in \Delta\} \in \ell^1(\Delta)$ where $\Delta \subset \mathbb{R}^n$. Then,

$$v(x) = \sum_{\alpha\in\Delta} c_\alpha e^{2\pi i \alpha\cdot x} = 0 \text{ for all } x \in \mathbb{R}^n \iff c_\alpha = 0 \ \forall \alpha\in\Delta$$

P/ If $v(x)=0$ and $\beta \in \Delta$

$$0 = \lim_{R\to\infty} \frac{1}{(2R)^n} \int_{[-R,R]^n} v(x) e^{-2\pi i \beta\cdot x} dx =$$

$$= \lim_{R\to\infty} \sum_{\alpha\in\Delta} c_\alpha \frac{1}{(2R)^n} \int_{[-R,R]^n} e^{2\pi i (\alpha-\beta)\cdot x} dx .$$

If $\alpha=\beta$ the last integral mean is 1. If $\alpha\neq\beta$,

$$\frac{1}{(2R)^n} \int_{[-R,R]^n} e^{2\pi i (\alpha-\beta)\cdot x} dx = \prod_{j=1}^n \left\{ \frac{1}{2R} \int_{-R}^{R} e^{2\pi i (\alpha_j-\beta_j) x_j} dx_j \right\}$$

Since $\alpha_j \neq \beta_j$ for at least one j

$$\frac{1}{2R} \int_{-R}^{R} e^{2\pi i (\alpha_j-\beta_j) x_j} dx_j = \frac{1}{2R} \frac{\operatorname{sen}(2\pi(\alpha_j-\beta_j)R)}{\pi(\alpha_j-\beta_j)} \xrightarrow{R\uparrow\infty} 0 .$$

Hence $c_\beta = 0$. ∎

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REMARK. The above lemma is the uniqueness of the "Fourier coefficients" of an almost-periodic function $f:\mathbb{R}^n \to \mathbb{C}$:

$\forall \varepsilon > 0$, there exists $T \equiv T_\varepsilon$ s.t.

$|f(x+T_\varepsilon) - f(x)| < \varepsilon$ for all $x \in \mathbb{R}^n$

THEOREM 5.2.

Assume L.I.C. $\{T_{C_p k} g_p : p \in P, k \in \mathbb{Z}^n\}$ is a PF $\iff$

$$\sum_{p \in P_\alpha} \frac{1}{|\det C_p|} \overline{\hat{g}_p(\xi)}\, \hat{g}_p(\xi+\alpha) = \delta_{\alpha,0} \quad \text{a.e. } \xi \in \mathbb{R}^n, \text{ for all}$$

$$\alpha \in \Lambda = \bigcup_{p \in P} C_p^{\sharp}(\mathbb{Z}^n) \qquad (*)$$

P/ $\Leftarrow$) Assuming $(*)$, if $f \in \mathcal{D}$, by lemma 5.5 $w_f(x) = \sum_{\alpha \in \Lambda} \hat{w}_f(\alpha) e^{2\pi i \alpha \cdot x}$ where $\hat{w}_f(\alpha) = \left(\int_{\mathbb{R}^n} \hat{f}(\xi) \overline{\hat{f}(\xi+\alpha)}\, d\xi\right) \delta_{\alpha,0}$

$\therefore$ $w_f(\alpha) = 0$ if $\alpha \neq 0$ and $\hat{w}_f(0) = \|f\|^2$. For $x = 0$

$$\sum_{p \in P} \sum_{k \in \mathbb{Z}^n} |\langle f, T_{C_p k} g_p \rangle|^2 = w_f(0) = \hat{w}_f(0) = \|f\|^2 .$$

$\Rightarrow$ Suppose now $w_f(x) \equiv N^2(f) = \|f\|^2$ (PF condition) for all $f \in L^2(\mathbb{R}^n)$. By lemma 5.5, if $f \in \mathcal{D}$, the function $z_f(x) = w_f(x) - \|f\|^2$ is continuous and coincides with its almost-periodic Fourier series, so that

$$\hat{z}_f(0) = \hat{w}_f(0) - \|f\|^2 \quad , \quad \hat{z}_f(\alpha) = \hat{w}_f(\alpha) \quad , \ \alpha \neq 0 .$$

Since $w_f(x) = \|f\|^2$ (PF condition), $z_f(x) = 0$. By lemma 5.6, $\hat{z}_f(\alpha) = 0$ for all $\alpha \in \Lambda$. Hence $\hat{w}_f(\alpha) = \delta_{\alpha,0} \|f\|^2$ for all $\alpha \in \Lambda$. By lemma 5.5

$$\int_{\mathbb{R}^n} \hat{f}(\xi) \overline{\hat{f}(\xi+\alpha)} \left(\sum_{p \in P_\alpha} \frac{1}{|\det C_p|} \overline{\hat{g}_p(\xi)}\, \hat{g}_p(\xi+\alpha) \right) d\xi = \delta_{\alpha,0} \|f\|^2$$

for all $f \in \mathcal{D}$. By the Lebesgue differentiation thm

$$\sum_{p \in P_\alpha} \frac{1}{|\det C_p|} \overline{\hat{g}_p(\xi)}\, \hat{g}(\xi+\alpha) = \delta_{\alpha,0} \quad \text{a.e. } \xi \in \mathbb{R}^n, \alpha \in \Lambda .$$

$\blacksquare$