WAVELETS AND GABOR FRAMES: A UNIFIED THEORY

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Recent developments in real and harmonic analysis

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1. DEFINICIONES, NOTACIONS Y EJEMPLOS

In a separable Hilbert space $\mathbb{H}$, $(\{\varphi_\alpha\}, \{\psi_\alpha\})$ is a <u>reproducing system</u> if for all $f \in \mathbb{H}$

$$f = \sum_{\alpha \in A} \langle f, \varphi_\alpha \rangle \psi_\alpha$$

with convergence in $\mathbb{H}$. The analysis of f is done with the $\{\varphi_\alpha\}$ and the reconstruction with the $\{\psi_\alpha\}$.

A family $\{\varphi_\alpha : \alpha \in A\}$ in $\mathbb{H}$ is a FRAME if there exists constants $0 < A \leq B < \infty$ such that

$$A \|f\|^2 \leq \sum_{\alpha \in A} |\langle f, \varphi_\alpha \rangle|^2 \leq B \|f\|^2 \quad , f \in \mathbb{H}.$$

If only the right hand side inequality holds we say that the system is BESSEL. If $A = B = 1$ the system is called a PARSEVAL FRAME.

- EXAMPLE 1. The system $\{g_{m,n}(x) = e^{2\pi i m x} g(x-n) : m, n \in \mathbb{Z}\}$ is an orthonormal basis for $L^2(\mathbb{R})$, thus a reproducing system and a frame

- EXAMPLE 2. Let ψ be defined by $\hat{\psi}(\xi) = \chi_{I_a}(\xi)$ where $I_a = [2a, a) \cup (a, 2a]$ and $a \leq 1/2$. The system

$$\{\psi_{j,k}(x) = 2^{j/2} \psi(2^j x - k) : j, k \in \mathbb{Z}\}$$

is a Parseval frame

Proof.

$$\sum_{j,k} |\langle f, \psi_{j,k} \rangle|^2 = \sum_{j,k} |\langle \hat{f}, \widehat{\psi_{j,k}} \rangle|^2 =$$

$$= \sum_{j,k}' \left| \int_{\mathbb{R}} \hat{f}(\xi)\, 2^{-j/2} \hat{\psi}(2^{-j}\xi)\, e^{-2\pi i 2^{-j}\xi \cdot k}\, d\xi \right|^2$$

$$= \sum_{j,k}' \left| \int_{I_a} \hat{f}(2^j\mu)\, 2^{j/2} \hat{\psi}(\mu)\, e^{-2\pi i \mu \cdot k}\, d\mu \right|^2$$

$$= \sum_{j,k}' 2^j \left| \int_{I_{1/2}} \hat{f}(2^j\mu)\, \chi_{I_a}(\mu)\, e^{-2\pi i \mu \cdot k}\, d\mu \right|^2$$

Since $\{e^{-2\pi i \mu \cdot k} : k \in \mathbb{Z}\}$ is an orthonormal basis for $L^2(I_{1/2})$, by Plancherel we obtain

$$\sum_{j,k}' |\langle f, \psi_{j,k}\rangle|^2 = \sum_j 2^j \int_{I_{1/2}} |\hat{f}(2^j\mu)\, \chi_{I_a}(\mu)|^2\, d\mu$$

$$= \sum_j \int_{\mathbb{R}} |\hat{f}(\xi)|\, \chi_{I_a}(2^{-j}\xi)\, d\xi = \|\hat{f}\|^2 = \|f\|^2$$

since $\displaystyle\sum_{j \in \mathbb{Z}} \chi_{I_a}(2^{-j}\xi) = 1$ for all $\xi \neq 0$.

- TRASLACIÓN: $T_y f(x) = f(x-y)$, $y \in \mathbb{R}^n$
- DILATACIÓN: for $A \in GL_n(\mathbb{R}^n)$,

$$D_A f(x) = |\det A|^{1/2} f(Ax)$$

- MODULATION: for $z \in \mathbb{R}^n$, $M_z f(x) = e^{2\pi i x \cdot z} f(x)$

LEMMA 1.1.

(1) $(T_y f)^\wedge = M_y \hat{f}$ (2) $(M_z f)^\wedge = T_z \hat{f}$

(3) $(D_A f)^\wedge = D_{(A^t)^{-1}} \hat{f}$ (4) $T_y M_z f = e^{-2\pi i z\cdot y} M_z T_y f$

(5) $D_A T_y f = T_{A^{-1}y} D_A f$ (6) $D_A M_y f = M_{A^t y} D_A f$

We will mainly deal with the following type of systems:

GABOR OR WEYL-HEISENBERG SYSTEMS on $L^2(\mathbb{R}^n)$:
for $\{g^1, \dots, g^L\} \subset L^2(\mathbb{R}^n)$ and $B, C \in GL_n(\mathbb{R})$

$$\mathcal{G} \equiv \mathcal{G}_{B,C}(\{g^1, \dots, g^L\}) =$$

$$= \{T_{Ck} M_{Bm} g^\ell : k \in \mathbb{Z}^n, m \in \mathbb{Z}^n, \ell = 1, \dots, L\}$$

WAVELET TYPE SYSTEMS in $L^2(\mathbb{R}^n)$:
for $\{\psi^1, \dots, \psi^L\} \subset L^2(\mathbb{R}^n)$ and $A \in GL_n(\mathbb{R})$

$$\mathcal{W} \equiv \mathcal{W}_A(\{\psi^1, \dots, \psi^L\}) =$$

$$= \{D_{A^j} T_k \psi^\ell : k \in \mathbb{Z}^n, j \in \mathbb{Z}, \ell = 1, \dots, L\}$$

R.J. Duffin, A.C. Shaffer, "A Class of nonharmonic Fourier series", Trans. Amer. Math. Soc 72(1952), 341-366.

2. RECONSTRUCTION WITH FRAMES

The purpose of this section is to show that any frame in a Hilbert space $\mathbb{H}$ generates a reproducing system for elements of $\mathbb{H}$. The result is easy if the frame is tight.

LEMMA 2.1.
If $\{\varphi_j\}_{j=1}^{\infty}$ is a tight frame with constant A we have
$$f = \frac{1}{A}\sum_{j=1}^{\infty} \langle f, \varphi_j \rangle \varphi_j, \quad \text{for all } f \in \mathbb{H}.$$

Proof: The condition of been a tight frame is
$$\sum_{j=1}^{\infty} |\langle f, \varphi_j \rangle|^2 = A \|f\|^2 \qquad \text{for all } f \in \mathbb{H}. \qquad (1)$$
Equality (1) implies that the sequence $S_N = \sum_{j=1}^{N} \langle f, \varphi_j \rangle \varphi_j$ of partial sums is a Cauchy sequence in $\mathbb{H}$. Thus, the series converges to an element $h \in \mathbb{H}$, that is
$$h = \sum_{j=1}^{\infty} \langle f, \varphi_j \rangle \varphi_j \qquad \text{in } \mathbb{H}. \qquad (2)$$

Using the polarization identity in $\mathbb{H}$ as well as in $\ell^2(\mathbb{N})$ it can be deduced from (1) that
$$A \langle f, g \rangle = \sum_{j=1}^{\infty} \langle f, \varphi_j \rangle \langle \varphi_j, g \rangle \qquad (3)$$
for all $f, g \in \mathbb{H}$. Thus, for all $g \in \mathbb{H}$ using (2) and (3) we obtain
$$\langle h, g \rangle = \langle \sum_{j=1}^{\infty} \langle f, \varphi_j \rangle \varphi_j, g \rangle = A \langle f, g \rangle$$
which implies $Af = h$

For a general frame $\{\varphi_j\}_{j=1}^{\infty}$ we define the ANALYSIS OPERATOR $A : \mathbb{H} \to \ell^2(\mathbb{N})$ by

$$Af = \{\langle f, \varphi_j \rangle\}_{j=1}^{\infty} .$$

The operator A is linear and bounded with $\|A\| \leq \sqrt{B}$. Its adjoint $A^* : \ell^2(\mathbb{N}) \longrightarrow \mathbb{H}$, which is called the RECONSTRUCTION OPERATOR is also bounded with norm less than $\sqrt{B}$ and is given by

$$A^*(\{c_k\}) = \sum_k c_k \varphi_k .$$

FRAME OPERATOR: $S = A^*A : \mathbb{H} \longrightarrow \mathbb{H}$

$$S(f) = \sum_k \langle f, \varphi_k \rangle \varphi_k \quad , \; f \in \mathbb{H}. \qquad (1)$$

The frame condition is equivalent to

$$A\langle f, f \rangle \leq \langle Sf, f \rangle \leq B\langle f, f \rangle \quad , f \in \mathbb{H} \qquad (2)$$

Hence, S is a positive linear operator that satisfies

$$AI \leq S \leq BI \qquad (3)$$

From (3) we deduce that S has an inverse s.t.

$$\frac{1}{B} I \leq S^{-1} \leq \frac{1}{A} I \qquad (4)$$

LEMMA 2.2.

If $\{\varphi_j\}$ is an $(A-B)$ frame on $\mathbb{H}$, then $\{\tilde{\varphi}_j = S^{-1}\varphi_j\}$ is an $(\frac{1}{B}, \frac{1}{A})$ frame on $\mathbb{H}$ (called the DUAL FRAME of $\{\varphi_j\}$)

PROOF. From (4), $\frac{1}{B}\|f\|^2 \le \langle S^{-1}f, f\rangle \le \frac{1}{A}\|f\|^2$. Enough to show

$$\langle S^{-1}f, f\rangle = \sum_j |\langle f, \tilde{\varphi}_j\rangle|^2 \quad , f \in \mathbb{H} \qquad (5)$$

Since $S = A^*A$ is self-adjoint, so is S^{-1}. Thus,

$$\sum_j |\langle f, \tilde{\varphi}_j\rangle|^2 = \sum_j |\langle f, S^{-1}\varphi_j\rangle|^2 = \sum_j |\langle S f, \varphi_j\rangle|^2$$

$$= \|A(S^{-1}f)\|_{\ell^2}^2 = \langle A(A^*A)^{-1}f, A(A^*A)^{-1}f\rangle$$

$$= \langle (A^*A)^{-1}f, f\rangle = \langle S^{-1}f, f\rangle \qquad \blacksquare$$

Defining $\langle f, g\rangle_* = \langle S^{-1}f, g\rangle$ we have an inner product on $\mathbb{H}$ and by (5)

$$\|f\|_*^2 = \sum_k |\langle f, \varphi_j\rangle_*|^2$$

Hence, $\{\varphi_j\}$ is a PARSEVAL FRAME on $\mathbb{H}$ with $\langle , \rangle_*$. By lemma 1.1.

$$f = \sum_j \langle f, \varphi_j\rangle_* \varphi_j = \sum_j \langle f, \tilde{\varphi}_j\rangle \varphi_j \quad , f \in \mathbb{H}. \quad (6)$$

The dual formula can be deduced from here:

$$\langle f, g\rangle = \langle \sum_j \langle f, \tilde{\varphi}_j\rangle \varphi_j, g\rangle = \sum_j \langle f, \tilde{\varphi}_j\rangle\langle \varphi_j, g\rangle$$

$$= \langle f, \sum_j \langle g, \varphi_j\rangle \tilde{\varphi}_j\rangle \quad \therefore \quad g = \sum_j \langle g, \varphi_j\rangle \tilde{\varphi}_j \quad \text{(weak)}$$

Convergence is also in the norm of $\mathbb{H}$ because $\{\varphi_j\}$ is Bessel with constant B and $\{\tilde{\varphi}_j\}$ is Bessel with constant $\frac{1}{A}$.

We have proved:

THEOREM 3.

If $\{\varphi_j\}$ is an (A,B) frame, $\{\tilde{\varphi}_j = S^{-1}\varphi_j\}$ is an $(\frac{1}{B}, \frac{1}{A})$ frame and

$$f = \sum_j \langle f, \tilde{\varphi}_j\rangle \varphi_j \quad , \quad f = \sum_j \langle f, \varphi_j\rangle \tilde{\varphi}_j \qquad (f \in \mathbb{H})$$

with convergence in $\mathbb{H}$.

This proves that any frame produces a reproducing system with its dual frame. The coefficients $\langle f, \tilde{\varphi}_j\rangle$ are not necessarily unique. But they have a minimality property.

PROPOSITION 4.

Let $\{\varphi_j\}$ be a frame on $\mathbb{H}$ and $f \in \mathbb{H}$. If $f = \sum_k c_k \varphi_k$ (in $\mathbb{H}$) with $c = \{c_k\} \in \ell^2$, then

$$\sum_j |c_j|^2 \geq \sum_j |\langle f, \tilde{\varphi}_j\rangle|^2$$

where $\{\tilde{\varphi}_j\}$ is the dual frame to $\{\varphi_j\}$. Equality holds iff $c_j = \langle f, \tilde{\varphi}_j\rangle$.

PROOF. With $a_j = \langle f, \tilde{\varphi}_j\rangle$ we have $f = \sum_j a_j \varphi_j$. Since S^{-1} is self-adjoint

$$\langle f, S^{-1}f\rangle = \sum_j a_j \langle \varphi_j, S^{-1}f\rangle = \sum_j a_j \langle S^{-1}\varphi_j, f\rangle = \sum_j |a_j|^2$$

On the other hand,

$$\langle f, S^{-1}f\rangle = \sum_j c_j \langle \varphi_j, S^{-1}f\rangle = \sum_j c_j \bar{a}_j = \langle c, a\rangle_{\ell^2}$$

Hence, $\|a\|_2^2 = \langle c, a\rangle_{\ell^2}$ and

$$\|c\|_{\ell^2}^2 = \|c - a + a\|_{\ell^2}^2 = \|c-a\|^2 + \|a\|^2 + \langle c-a, a\rangle + \langle a, c-a\rangle$$

$$= \|c-a\|^2 + \|a\|^2 \geq \|a\|^2$$

and equality holds iff. $c = a$. ∎

3. FRAMES GENERATED BY A SINGLE FUNCTION.

(or a finite number of functions)

Q.1. Does there exist $g \in L^2(\mathbb{R}^n)$ s.t. $\{T_k g : k \in \mathbb{Z}^n\}$ is a frame for $L^2(\mathbb{R}^n)$?

Q.2. Does there exist $g \in L^2(\mathbb{R}^n)$ s.t. $\{M_k g : k \in \mathbb{Z}^n\}$ is a frame for $L^2(\mathbb{R}^n)$?

Q.3. Does there exist $g \in L^2(\mathbb{R}^n)$ s.t. $\{D_A^j g : j \in \mathbb{Z}\}$ is a frame for $L^2(\mathbb{R}^n)$?

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Suppose $\{T_k g : k \in \mathbb{Z}^n\}$ is a frame for $L^2(\mathbb{R}^n)$. Then

$$A\|f\|^2 \le \sum_k |\langle f, T_k g\rangle|^2 \le B\|f\|^2 \quad , f \in L^2(\mathbb{R}^n) \qquad (1)$$

From (1), for any $u \in [0,1)^n$

$$A\|f\|^2 \le \sum_k |\langle f, T_{k+u} g\rangle|^2 \le B\|T_{-u} f\|^2 = \|f\|^2 B \qquad (2)$$

Integrating (2) in $[0,1)^n$

$$A\|f\|^2 \le \int_{[0,1)} \sum_k |\langle f, T_{u+k} g\rangle|^2 du = \int_{\mathbb{R}^n} |\langle f, T_y g\rangle|^2 dy$$

$$\le B\|f\|^2 \quad , f \in L^2(\mathbb{R}^n) \qquad (3)$$

But

$$\langle f, T_y g\rangle \overset{\text{Plan}}{=} \langle \hat{f}, (T_y g)^\wedge\rangle = \langle \hat{f}, M_{-y}\hat{g}\rangle =$$

$$= \int_{\mathbb{R}^n} \hat{f}(\xi)\overline{\hat{g}(\xi)} e^{2\pi i y\cdot\xi} d\xi = (\hat{f}\cdot\overline{\hat{g}})^\vee(y)$$

From (3)

$$A\|\hat{f}\|^2 \le \int_{\mathbb{R}^n} |\hat{f}(\xi)\overline{\hat{g}(\xi)}|^2 d\xi \le B\|\hat{f}\|^2 \quad , f \in L^2(\mathbb{R}^n)$$

This implies

$$A \le |\hat{g}(\xi)| \le B \quad \text{a.e. } \xi \in \mathbb{R}^n$$

which shows that $g \notin L^2(\mathbb{R}^n)$

$$\boxed{\text{There are no frames in } L^2(\mathbb{R}^n) \text{ of the form } \{T_k g : k \in \mathbb{Z}^n\}}$$

REMARK 1. Similar argument shows that there are no frames in $L^2(\mathbb{R}^n)$ of the form $\{T_{Ck} g : k \in \mathbb{Z}^n\}$ and $C \in GL_n(\mathbb{R})$. Replace $[0,1)^n$ by $C[0,1)^n$

REMARK 2. [O. Christensen, B. Deng, C. Heil] "Density of Gabor frames (1999) showed that if $\Gamma \subset \mathbb{R}$ is denumerable, then $\{T_a g : a \in \Gamma\}$ is not a frame.

If $\{M_k g : k \in \mathbb{Z}\}$ is a frame for $L^2(\mathbb{R}^n)$, by Plancherel's theorem, $\{(M_k g)^\wedge : k \in \mathbb{Z}\} = \{T_k \hat{g} : k \in \mathbb{Z}\}$ is a frame for $L^2(\mathbb{R}^n)$. Thus:

$$\boxed{\text{There are no frames in } L^2(\mathbb{R}^n) \text{ of the form } \{M_k g : k \in \mathbb{Z}^n\}}$$

Suppose $\{D_A^j g : j \in \mathbb{Z}\}$ is a frame for $L^2(\mathbb{R}^n)$, $A \in GL_n(\mathbb{R})$. Start with $A = 2$ $(n=1)$. By example 1 of section 1, if $\hat{\psi} = \chi_{I_a}$, $I_a = [-2a, a) \cup (a, 2a]$ $(a \le \frac{1}{2})$, then $\{D_{2^j} T_k \psi : j \in \mathbb{Z}, k \in \mathbb{Z}\}$ is a Parseval frame. Hence,

$$\|g\|^2 = \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} |\langle g, D_{2^j} T_k \psi\rangle|^2$$ Change vars

$$= \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} |\langle D_{2^{-j}} g, T_k \psi\rangle|^2 \geq$$ $\{D_{2^j} g : j\in\mathbb{Z}\}$ frame

$$\geq \sum_{k\in\mathbb{Z}} A \|T_k \psi\|^2 = \sum_{k\in\mathbb{Z}} A \|\psi\|^2 = \infty$$

> There are no frames in $L^2(\mathbb{R}^n)$ of the form
> $\{D_{2^j} g : j \in \mathbb{Z}\}$.

REMARK. For $A \in Gl_n(\mathbb{R})$ expansive, [Dai, Larson, Speegle] "Wavelet sets in $\mathbb{R}^n$" (1997), showed that there exists $\psi \in L^2(\mathbb{R}^n)$ s.t. $\{D_{A^j} T_k \psi : j \in \mathbb{Z}, k \in \mathbb{Z}^n\}$ is an o.n. wavelet (hence, a PF). Thus, the above result is true on $\mathbb{R}^n$ for expansive matrices.

Are there frames of the form

1. $\{T_{Ck} M_{Bm} g : k, m \in \mathbb{Z}^n\}$
2. $\{M_{Bm} T_{Ck} g : k, m \in \mathbb{Z}^n\}$
3. $\{D_{A^j} T_{Ck} g : j \in \mathbb{Z}, k \in \mathbb{Z}^n\}$
4. $\{T_{Ck} D_{A^j} g : j \in \mathbb{Z}, k \in \mathbb{Z}^n\}$
5. $\{D_{A^j} M_{Bm} g : j \in \mathbb{Z}, m \in \mathbb{Z}^n\}$
6. $\{M_{Bm} D_{A^j} g : j \in \mathbb{Z}, m \in \mathbb{Z}^n\}$

for $g \in L^2(\mathbb{R}^n)$, $A, B, C \in Gl_n(\mathbb{R})$?

FRAMES OF TYPE ①. - GABOR FRAMES -

a) If $g \in W(\mathbb{R}^n)$ - Wiener space:

$$\|g\|_W = \sum_{k \in \mathbb{Z}^n} \sup_{x \in [0,1]^n} |g(x+k)| < \infty$$

and $|\det C|$, $|\det B|$ small, then $\{T_{Ck} M_{Bm} g : k \in \mathbb{Z}^n, m \in \mathbb{Z}^n\}$ is a frame ([D. Walnut, "Continuity properties of the Gabor frame operator" (1992) for the case $C = \gamma I$, $B = \beta I$; for general $A, B \in Gl_n(\mathbb{R})$, D. Vera (2004)).

b) [M.A. Rieffel], "Von Neumann algebras associated with pair of lattices in Lie groups" (1995): if $\gamma \cdot \beta > 1$ the family $\{T_{\gamma k} M_{\beta m} g : k, m \in \mathbb{Z}\}$ is not a frame $(\gamma, \beta > 0)$. For matrices, [E. Hernández, Labate, Weiss, 2002].

c) If $\gamma \cdot \beta = 1$, the functions $g \in L^2(\mathbb{R})$ for which $\{T_{\gamma k} M_{\beta m} g : k, m \in \mathbb{Z}\}$ is a frame must satisfy

$$\left(\int_{\mathbb{R}} x^2 |g(x)|^2 dx\right)\left(\int_{\mathbb{R}} \xi^2 |\hat{g}(\xi)|^2 d\xi\right) = \infty$$

(Balian-Low) [Balian] "Un principe d'incertitude fort en théorie du signal ou en mécanique quantique" (1981)

FRAMES OF TYPE ②

Since

$$M_{Bm} T_{Ck} g(x) = e^{2\pi i Bm \cdot x} g(x - Ck) =$$

$$= e^{2\pi i Bm (x - Ck)} e^{2\pi i Bm \cdot Ck} g(x - Ck) =$$

$$= e^{2\pi i Bm \cdot Ck} T_{Ck} M_{Bm} g(x)$$

and $|e^{2\pi i Bm \cdot Ck}| = 1$, ② is a frame $\Leftrightarrow$ ① is a frame

FRAMES OF TYPE ③ & ⑤

These are wavelet type systems. These type of frames are known to exists for many matrices A (expansive). The general case of finding all A and C s.t. ③ is a frame ~~is~~ ~~still~~ for some $g \in L^2(\mathbb{R}^n)$ is still unsolved.

By taking Fourier transforms

$$(D_A^j M_{Bm} g)^\wedge = D_{(A^t)^{-1}}^j T_{Bm} \hat{g}$$

frames of type ⑤ have the same properties than frames of type ④.

FRAMES OF TYPE ④ and ⑥

Taking the Fourier transform, existence of frames of the form ④ is equivalent to existence of frames of the form ⑥. Frames of this type do not exist (at least for many matrices A) [P. Gressman, D. Labate, G. Weiss, E. Wilson], (2001).

THEOREM 3.1. (Gressman, Labate, Weiss, Wilson)

There are no frames of the ~~type~~ ~~④ and ⑥ when~~ form

$$\{T_k D_A^j g : k \in \mathbb{Z}^n, j \in \mathbb{Z}\} \text{ or } \{M_m D_A^j g : m \in \mathbb{Z}^n, j \in \mathbb{Z}\}$$

when $A \in Gl_n(\mathbb{R})$, $|\det A| \neq 1$ and $\exists\, X \subset \mathbb{R}^n$, $0 < |X| < \infty$ s.t. $\{A^j(X)\}_{j \in \mathbb{Z}}$ disjoint.

P/. Suppose $\{T_k D_A^j g : k \in \mathbb{Z}^n, j \in \mathbb{Z}\}$ is a frame. Then

$$A_1 \|f\|^2 \leq \sum_j \sum_k |\langle f, T_k D_A^j g\rangle|^2 \leq B_1 \|f\|^2, \quad f \in L^2(\mathbb{R}^n) \quad (1)$$

With $f = g$ we obtain $\|g\|^4 = |\langle g, g\rangle|^2 \leq B_1 \|g\|^2 \therefore$

$\|g\| \le \sqrt{B_1}$. For $x \in \mathbb{R}^n$ define

$$\omega_f(x) = \sum_j \sum_k |\langle T_x f, T_k D_A^j g\rangle|^2$$

which is $\mathbb{Z}^n$-periodic. From (1)

$$A_1 \|f\| \le \omega_f(x) \le B_1 \|f\| \qquad , f \in L^2(\mathbb{R}^n) \qquad (2)$$

Integrating on $[0,1)^n$ we deduce:

$$\int_{[0,1)^n} \omega_f(x)\,dx = \sum_j \sum_k \int_{[0,1)^n} |\langle f, T_{k-x} D_A^j g\rangle|^2\,dx$$

$$= \sum_j \sum_k \int_{[0,1)^n - k} |\langle f, T_{-y} D_A^j g\rangle|^2 = \sum_j \int_{\mathbb{R}^n} |\langle f, T_{-y} D_A^j g\rangle|^2$$

$$\underset{\text{Plancherel}}{=} \sum_j \int_{\mathbb{R}^n} |\langle \hat{f}, M_y D_{(A^t)^{-j}} \hat{g}\rangle|^2\,dy$$

$$= \sum_j \int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} \hat{f}(\xi)\, |\det A|^{-j/2}\, \overline{\hat{g}((A^t)^{-j}\xi)}\, e^{-2\pi i y\cdot\xi}\,d\xi \right|^2 dy$$

$$\underset{\text{Plan}}{=} \sum_j |\det A|^{-j} \int_{\mathbb{R}^n} |\hat{f}(\xi)|\,|\hat{g}((A^t)^{-j}\xi)|^2\,d\xi \qquad (3)$$

From (2) and (3) we obtain

$$A_1 \le \sum_j |\det A|^{-j} |\hat{g}((A^t)^{-j}\xi)|^2 \le B_1 \qquad \text{a.e. } \xi \in \mathbb{R}^n \quad (4)$$

For all $m \in \mathbb{Z}$

$$A_1 |X| \le \sum_j \int_X |\det A|^{-j} |\hat{g}((A^t)^{-j+m}\xi)|^2\,d\xi \qquad \underline{\underline{-j+m=\ell}}$$

$$= \sum_\ell \int_X |\det A|^{\ell-m} |\hat{g}((A^t)^{\ell}(\xi))|^2\,d\xi \qquad \underline{\underline{(A^t)^\ell \xi = \eta}}$$

$$= \sum_\ell \int_{(A^t)^\ell(X)} |\det A|^{-m} |\hat{g}(\eta)|^2\,d\eta$$

$$\le \int_{\mathbb{R}^n} |\det A|^{-m} |\hat{g}(\eta)|^2\,d\eta \le B_1 |\det A|^{-m}$$

letting $m \to \infty$ or $-\infty$, this is a contradiction since $|\det A| \ne 1$ ∎