4.1: Functions

The concept of **"function"** is one of the most important in mathematics.
Definition: A ***function*** is a correspondence between a set of input values (independent variable) and a set of output values (dependent variable) in such a way that for each input value, there is exactly on corresponding output value.

In other words, a ***function*** is an equation where if an *x* value is plugged into the equation will yield exactly one *y value* out of the equation.

Functions are denoted by *f(x).* The notation *f(x)* means that we have a **function** named *f* and the **variable** in that **function** is *x*. (i.e " *f(x)*" emphasizes that the value of y depends on the value of *x*).

Example 1: Determine which of the following equations are functions and which are not functions.

a) $y = 5x + 1$

b) $y = x^2 + 1$

c) $y^2 = x + 1$

Solution

The definition of function is saying that if we take all possible values of *x* and plug them into the equation and solve for *y* we will get exactly one value for each value of *x*.

(a) $y = 5x + 1$

We need to show that no matter what *x value* we plug into the equation and solve for *y*, we will only get a single value of *y*. Note that the value of *y* will probably be different for each value of *x*, although it does not have to be.

Start by plugging in some values of *x* and see what happens.

$x = -4$: $\quad y = 5(-4) + 1 = -20 + 1 = -19$

$x = 0$: $\quad y = 5(0) + 1 = 0 + 1 = 1$

$x = 10$: $\quad y = 5(10) + 1 = 50 + 1 = 51$

So, for each of these value of *x* we got a single value of *y* out of the equation. Now, this is not sufficient to claim that this is a function. In order to officially prove that this is a function we need to show that this will work no matter which value of *x* we plug into the equation.

Of course we cannot plug all possible value of *x* into the equation. That just is not physically possible. However, let us go back and look at the ones that we did plug in. For each *x*, upon plugging in, we first multiplied the *x* by 5 and then added 1 to it. Now, if we multiply a number by 5 we will get a single value from the multiplication. Likewise, we will only get a single value if we add 1 to a number. Therefore, based on the operations involved when plugging *x* into the equation we will only get a single value of *y* out of the equation.

So, this equation is a function.

b) $y = x^2 + 1$ **(Please see class notes).**

c) $y^2 = x + 1$

As we have done with the previous two equations let us plug in a couple of value of *x*, solve for *y* and see what we get.

$x = 3:$	$y^2 = 3 + 1 = 4$	$\Rightarrow$	$y = \pm 2$
$x = -1:$	$y^2 = -1 + 1 = 0$	$\Rightarrow$	$y = 0$
$x = 10:$	$y^2 = 10 + 1 = 11$	$\Rightarrow$	$y = \pm\sqrt{11}$

Now, remember that we are solving for *y* and so that means that in the first and last case above we will actually get two different *y* values out of the *x* and so this equation is NOT a function.

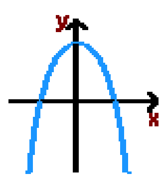
Note that we can have values of *x* that will yield a single *y* as we have seen above, but that does not matter. If even one value of *x* yields more than one value of *y* upon solving the **equation will not be a function**.

** **Vertical line test:-** is used to determine whether the graph is that of a function or not.

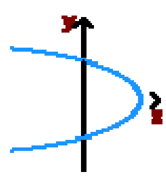
- Any vertical line can intersect the graph of a function **ONLY once.**

Example 2**:** Use the vertical line test to determine which of the graphs are graphs of functions.

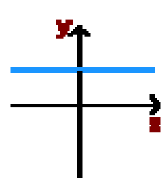
a)



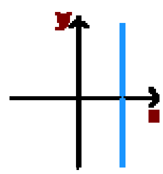
b)



c)



d)



** **Domain of functions:**
The **domain** of a function is the set of all input values (*x*-values) for which the function will produce a real number.
Remember that we have restrictions on denominators and square roots.

Example 3: Determine the domain of each of the following functions.

(a) $g(x)=\dfrac{x+3}{x^2+3x-10}$

(b) $f(x)=\sqrt{5-3x}$

(c) $h(x)=\dfrac{\sqrt{7x+8}}{x^2+4}$

Solution:
The domains for these functions are all the values of *x* for which we **do not have division by zero or the square root of a negative number.** If we remember these two ideas finding the domains will be easy.

(a) $g(x)=\dfrac{x+3}{x^2+3x-10}$
In this case there are no square roots so we do not need to worry about the square root of a negative number. There is however a possibility that we will have a division by zero. To determine the values of *x* that will yield a zero denominator, we will need to set the denominator equal to zero and solve.

$$x^2+3x-10=(x+5)(x-2)=0 \qquad x=-5,\ \ x=2$$

So, we will get division by zero if we plug in $x=-5,$ or $x=2$. That means that we need to avoid those two numbers. However, all the other values of *x* will work since they do not give division by zero. Then the domain is,

Domain : All real numbers except $x=-5,$ or $x=2$

(b) $f(x)=\sqrt{5-3x}$
In this case we have a square root in the problem and we will need to worry about taking the square root of a negative numbers.

This one is going to work a little differently from the previous part. In that part we determined the value(s) of *x* to avoid. In this case it will be easier to directly get the domain. To **avoid square roots of negative numbers** all that we need to do is set

$$5-3x\ge 0$$

This is a fairly simple linear inequality that we should be able to solve at this point.

$$5 \geq 3x \qquad \Rightarrow \qquad x \leq \frac{5}{3}$$

Then the domain of this function is,

$$\text{Domain}: \; x \leq \frac{5}{3}$$

(c) $h(x) = \dfrac{\sqrt{7x+8}}{x^2+4}$

In this case we have a fraction, but notice that the denominator will never be zero for any real number since x^2 is guaranteed to be positive or zero and adding 4 onto this will mean that the denominator is always at least 4. In other words, the denominator will not ever be zero. So, all we need to do then is worry about the square root in the numerator.

To do this we will set,

$$7x + 8 \geq 0$$
$$7x \geq -8$$
$$x \geq -\frac{8}{7}$$

Now, we can actually plug in any value of *x* into the denominator, however, since we have got the square root in the numerator we have to make sure that all *x*'s satisfy the inequality above to avoid problems. Therefore, the domain of this function is

$$\text{Domain}: \; x \geq -\frac{8}{7}$$

** **The Algebra of Functions:** Given two functions *f* and *g*, the **sum, difference, product,** and **quotient** of *f* and *g* are defined as follows.

$$(f+g)(x) = f(x) + g(x)$$
$$(f-g)(x) = f(x) - g(x)$$
$$(fg)(x) = f(x) \cdot g(x)$$
$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}, g(x) \neq 0$$

Example 4: Given $f(x) = -x^2 + 3x + 2$ and $g(x) = 2x - 1$, find the following:

a) $(f+g)(x)$

b) $(f-g)(x)$
c) $(fg)(x)$
d) $\left(\frac{f}{g}\right)(x)$

***Solution*:**

a) From the definition of the sum of two functions,
$(f+g)(x)=f(x)+g(x)=-x^2+3x+2+2x-1=-x^2+5x+1$; this means add the two functions by combining like terms.

b) $(f-g)(x)=-x^2+3x+2-(2x-1)=-x^2+3x+2-2x+1=-x^2+x+3$

c) $(fg)(x)=f(x).g(x)=(-x^2+3x+2)(2x-1)=-2x^3+x^2+6x^2-3x+4x-2$
$=-2x^3+7x^2+x-2$

d) $\left(\frac{f}{g}\right)(x)=\frac{f(x)}{g(x)}=\frac{-x^2+3x+2}{2x-1}$

** **Evaluating Functions:**

Evaluate: an algebraic substitution of numbers for variables in an expression. Once the substitutions are completed, follow the operation expression symbols to find the numerical value of the expression.

Example: Evaluate 3*x* when *x* = 4:

3(4) = 12

Example:

a) If $f(x)=2x+4$, find $f(2)$.

The notation $f(2)$ means that everywhere we see *x* in the function, we are going to replace it with 2. This gives :
$f(2)=2(2)+4=4+4=8$

So, $f(2)=8$

b) If $f(x)=3x^2-4x-1$. Find $f(-1)$ and $f(3)$.

To find $f(-1)$ we will substitute -1 into the <u>function</u> everywhere there is an *x*. This will give us :
$f(-1)=3(-1)^2-4(-1)-1=3(1)+4-1=6$
So, $f(-1)=6$

We use a similar process to find $f(3)$ to find

$f(3)=3(3)^2-4(3)-1=3(9)-12-1=14$

So, $f(3)=14$

When evaluating functions, we may not always be asked to evaluate the function at a particular numerical value.

c) If $f(x)=x^2-2x$. Find $f(b)$ and $f(a-5)$.

Although we are now asked to work with variables instead of numbers, the process is exactly the same. To find $f(b)$, everywhere we see an *x*, we substitute the letter *b*. Likewise, to find $f(a-5)$, substitute $a-5$ the entire quantity everywhere there is an *x*.

$f(b)=(b)^2-2(b)=b^2-2b$

$f(a-5)=(a-5)^2-2(a-5)=(a^2-10a+25)-2a+10=a^2-12a+35$

Composition of Functions:

Given two functions $f(x)$ and $g(x)$ we have the following two definitions.

1. The **composition** of $f(x)$ and $g(x)$ (note the order here) is denoted by,

   $$(fog)(x)=f[g(x)]$$

2. The **composition of** $g(x)$ and $f(x)$ (again, note the order) is denoted by,

   $$(gof)(x)=g[f(x)]$$

We need to note a couple of things here about function composition. First this is **NOT** multiplication, regardless of what the notation may suggest to you.

Second, the order we have listed the two functions is very important since more often than not we will get different answers depending on the order we have listed them.

Finally, function composition is really nothing more than **function evaluation**. All we are really doing is plugging the second function listed into the first function listed. In the definitions we used $[\ ]$ for the function evaluation instead of the standard $(\ )$ to avoid confusion with too many sets of parenthesis, but the evaluation will work the same.

Example: Given $f(x) = -x^2 + 3x + 2$ and $g(x) = 2x - 1$, find the following:

a) $(f o g)(x)$

Solution*:* Now, for function composition all you need to remember is that we are going to plug the second function listed into the first function listed. If you can remember that you should always be able to write down the basic formula for the composition.

Here is this function composition.

$$(f o g)(x) = f[g(x)]$$

$$= f[2x - 1]$$

Now, notice that since we have got an expression for $g(x)$ we went ahead and plugged that in first. Also, we did this kind of function evaluation in the previous **section.** Let us finish this problem out

$$\begin{aligned}(f o g)(x) &= f[g(x)] \\ &= f[2x - 1] \\ &= -(2x-1)^2 + 3(2x-1) + 2 \\ &= -(4x^2 - 4x + 1) + 6x - 3 + 2 \\ &= -4x^2 + 4x - 1 + 6x - 3 + 2 \\ &= -4x^2 + 10x - 2\end{aligned}$$

Notice that this is very different from the multiplication! **Remember that function composition is NOT function multiplication.**

b) $(g o f)(x)$

Solution*:* Proceed as in the previous part.

$$\begin{aligned}(g o f)(x) &= g[f(x)] \\ &= g[-x^2 + 3x + 2] \\ &= 2(-x^2 + 3x + 2) - 1 \\ &= -2x^2 + 6x + 4 - 1 \\ &= -2x^2 + 6x + 3\end{aligned}$$

Notice that this is NOT the same answer as that from the second part. In most cases the order in which we do the function composition will give different answers.

** **Difference Quotient:** Combining functions is a technique that is often used in calculus. For example, it is used in calculating the **difference quotient** of a function *f*, which is an expression in the form $\frac{f(x+h)-f(x)}{h}, h \neq 0.$

Example: Compute $\frac{f(x+h)-f(x)}{h}, h \neq 0$, for $f(x)=2x^2+1$

Solution: First, compute each component by step:

$$
\begin{aligned}
f(x+h) &= 2(x+h)^2+1 \\
&= 2\left(x^2+2xh+h^2\right)+1 \\
&= 2x^2+4xh+2h^2+1
\end{aligned}
$$

Next,

$$
\begin{aligned}
f(x+h)-f(x) &= 2x^2+4xh+2h^2+1-\left(2x^2+1\right) \\
&= 4xh+2h^2
\end{aligned}
$$

Finally,

$$\frac{f(x+h)-f(x)}{h}=\frac{4xh+2h^2}{h}=4x+2h$$

Example: Compute $\frac{f(x+h)-f(x)}{h}, h \neq 0$, for $f(x)=-x^2+4$

Also try the assigned HW problems.