

# FOX RE-EMBEDDING AND BING SUBMANIFOLDS

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ABSTRACT. Let  $M$  be an orientable closed connected 3-manifold, and let  $Y$  be a connected compact 3-manifold. We show that the following two statements are equivalent: (i)  $Y$  can be embedded in  $M$  so that the closure of the complement of the image of  $Y$  is a union of handlebodies; and (ii)  $Y$  can be embedded in  $M$  so that every knot in  $M$  can be isotoped to lie within the image of  $Y$ .

Our result can be regarded as a common generalization of Fox's re-embedding theorem [Fox48] and Bing's characterization of 3-sphere [Bin58], as well as more recent results of Hass and Thompson [HT89] and Kobayashi and Nishi [KN94].

## 1. INTRODUCTION

This paper presents a common generalization of two well-known classical theorems regarding the 3-sphere. The first is a theorem of Fox, published in 1948, and it is sometimes referred to as the Fox re-embedding theorem:

**Theorem 1** [Fox48]. *Every compact connected 3-submanifold  $Y$  of the 3-sphere can be re-embedded in the 3-sphere so that the closure of the complement of the image of  $Y$  is a union of handlebodies.*

The second is a theorem of Bing, published in 1958, and it gives a characterization of the 3-sphere:

**Theorem 2** [Bin58]. *A closed connected 3-manifold  $M$  is homeomorphic to the 3-sphere if and only if every knot in  $M$  can be isotoped to lie within a 3-ball  $B$  in  $M$ .*

Each of the above theorems has been generalized and refined over decades. The Fox re-embedding theorem was extended by Scharlemann and Thompson in [ST05], where they proved a more general re-embedding theorem for irreducible non-Haken 3-manifolds and also refined the re-embedding procedure so that the re-embedded submanifold and its complement are aligned in a suitable sense; see also [MT89] and [Sch92] for related results.

Bing's theorem was followed by many generalizations where the assumption was weakened in a hope to prove Poincaré Conjecture; see

for example, [McM61], [Mye78], [Mye82], [GM85]. In another direction, Hass and Thompson gave an analogous characterization of lens spaces [HT89]: a closed connected orientable 3-manifold  $M$  is homeomorphic to a lens space if and only if every knot in  $M$  can be isotoped to lie within a solid torus  $V$  in  $M$ . They also conjectured the higher genus generalization: a closed connected orientable 3-manifold  $M$  admits a genus  $g$  Heegaard splitting if and only if every knot in  $M$  can be isotoped to lie within a genus  $g$  handlebody  $V$  in  $M$ . This conjecture was proved by Kobayashi and Nishi in [KN94].

The properties of submanifolds that played crucial roles in the theorems above can be studied in a more general context of an arbitrary closed connected orientable 3-manifold  $M$  and a compact connected 3-submanifold  $Y$ . Let us say  $Y$  is a *Fox submanifold* of  $M$  if the closure of the complement of  $Y$  is a union of handlebodies and balls, and say  $Y$  is a *Bing submanifold* of  $M$  if every knot in  $M$  can be isotoped to lie within  $Y$ . Our main theorem is the following:

**Theorem 3.** *Let  $M$  be a closed connected orientable 3-manifold. Then, a compact connected 3-manifold can be embedded in  $M$  as a Fox submanifold if and only if it can be embedded in  $M$  as a Bing submanifold.*

It is not hard to see that the theorem indeed specializes to some of the known results discussed above. Also as immediate consequences of the theorem, we obtain the following corollaries.

**Corollary 4.** *A closed connected orientable 3-manifold  $M$  admits a genus  $g$  Heegaard splitting if and only if there exists a closed connected orientable surface  $F$  in  $M$  with genus  $g$  such that every knot in  $M$  can be isotoped to lie within a regular neighborhood of  $F$ .*

**Corollary 5.** *A closed connected orientable 3-manifold  $M$  admits a cross-cap genus  $g$  one-sided Heegaard splitting if and only if there exists a closed connected non-orientable surface  $F$  in  $M$  with cross-cap genus  $g$  such that every knot in  $M$  can be isotoped to lie within a regular neighborhood of  $F$ .*

Here is an outline of the paper. We review some basic notions of 3-manifolds in §2 and collect some facts about irreducible knots and simple knots in §3. Next, we study the genera of Heegaard splittings obtained by a process known as *amalgamation* in §4; the notion of *amalgamated Heegaard genus* introduced in this section plays an essential role in the proof of our main theorem. The main body of the proof is given in §5 as propositions involving the notion of *characteristic knots*. In §6, we complete the proof of the main theorem and discuss some of the results cited in this introduction; we will also explain why the main

theorem cannot be strengthened by replacing isotopy in the definition of Bing submanifold by homotopy. Indeed, our main theorem is not a consequence of the resolution of Poincaré Conjecture.

## 2. PRELIMINARIES

Throughout the paper, we assume that every 3-manifold is orientable and, unless otherwise stated, compact. We work with the piecewise linear category, and all embeddings are assumed to be locally flat.

A connected component of a manifold  $M$  (or, more generally, a complex) will simply be called a *component*, and we write  $|M|$  for the number of components. For a submanifold (or a subcomplex)  $X$  of a manifold (or a complex)  $M$ , we write  $N(X, M)$  and  $\overline{N}(X, M)$  respectively for an open regular neighborhood and a closed regular neighborhood of  $X$  with respect to the topology of  $M$ . We denote  $E(X, M) := M - N(X, M)$ ; when the ambient space  $M$  is clear from the context, we may simply write  $E(X)$  for  $E(X, M)$ .

A *link*  $L$  in a 3-manifold  $M$  is an embedded closed 1-manifold, and it is assumed to be non-empty unless otherwise stated. A *knot* is a link with one component. The *exterior* of a link  $L$  is defined to be  $E(L) = M - N(L)$ .

We call a non-empty 2-manifold a *surface*. If  $F$  is a closed connected orientable surface, the *genus* of  $F$  will be denoted by  $g(F)$  as usual. If  $F$  is a closed disconnected orientable surface, we write  $g(F)$  and  $g_{\max}(F)$  for the *total genus* and the *maximal component-wise genus* respectively. A simple closed curve on a closed surface  $F$ , i.e. a closed connected 1-submanifold of  $F$ , is said to be *essential* if it does not bound a disk on  $F$ , and it is said to be trivial otherwise.

Whenever we speak of surfaces in 3-manifolds, they are assumed to be embedded unless otherwise stated. If a surface  $F$  *separates* a 3-manifold  $M$  into  $Y_1$  and  $Y_2$ , i.e.  $M = Y_1 \cup Y_2$  and  $Y_1 \cap Y_2 = F$ , we write  $M = Y_1 \cup_F Y_2$ . An orientable surface  $F$  in a 3-manifold  $M$  is said to be *compressible* if either (i) there exists a component  $F^*$  of  $F$  and a 3-ball  $B$  in  $M$  such that  $B \cap F = \partial B = F^*$ , or (ii) there exists an embedded disk  $D$  in  $M$ , called a *compression disk* for  $F$ , such that  $\partial D$  is an essential simple closed curve on  $F$  and  $D \cap F = \partial D$ ;  $F$  is said to be *incompressible* otherwise. A 3-manifold  $M$  is said to be *irreducible* if every 2-sphere in  $M$  is compressible, and it is said to be *atoroidal* if every incompressible 2-torus in  $M$  is  $\partial$ -parallel.

In this paper, we will use a somewhat contemporary definition of compression bodies; more exposition based on this definition of compression bodies can be found in, for example, [HS01] or [SSS05]. A

*compression body*  $W$  is a connected 3-manifold obtained from a 3-ball  $B$  or  $F \times I$ , where  $F$  is a closed orientable surface and  $I = [0, 1]$ , by attaching some number (possibly 0) of 1-handles on  $\partial B$  or  $F \times \{1\}$ . We define  $\partial_- W := F \times \{0\}$  and  $\partial_+ W := \partial W - \partial_- W$ . A compression body is called a *handlebody* if  $\partial_- W = \emptyset$ . Note that, in this definition,  $\partial_+ W$  is allowed to be a sphere, and  $\partial_- W$  is allowed to have a sphere component. In a usual definition of compression body,  $\partial W$  must not have a sphere components.

A *spine* of a compression body  $W$  is an embedded graph  $\mathbf{A}$  such that  $E(\mathbf{A} \cup \partial_- W, W) \cong \partial_+ W \times I$ . A spine  $\mathbf{A}$  is said to be *minimal* if  $\mathbf{A}$  is a collection of properly embedded arcs. A *disk system*  $\mathbf{D} \subset W$  of a compression body  $W$  is a pairwise disjoint and pairwise non-parallel (possibly empty) collection of compression disks for  $\partial_+ W$  such that  $E(\mathbf{D}, W)$  is homeomorphic to a union of  $\partial_- W \times I$  and some 3-balls. A disk system  $\mathbf{D}$  is said to be *minimal* if (i)  $W$  is a handlebody and  $E(\mathbf{D}, W)$  is homeomorphic to a 3-ball, or (ii)  $\partial_- W \neq \emptyset$  and  $E(\mathbf{D}, W)$  is homeomorphic to  $\partial_- W \times I$ .

Given a minimal disk system  $\mathbf{D}$  of a compression body  $W$ , we can regard  $\overline{N}(\mathbf{D}, W)$  as a union of 1-handles. If  $W$  is a handlebody, we obtain a minimal spine  $\mathbf{A}$  of  $W$  by extending the core arcs of these 1-handles radially in  $E(\mathbf{D}, W) \cong B^3$  until they meet at the center. If  $W$  is not a handlebody, we obtain a minimal spine of  $W$  by extending the core arcs vertically in  $E(\mathbf{D}, W) \cong \partial_- W \times I$  until they reach  $\partial_- W$ . In either case, the spine  $\mathbf{A}$  obtained from  $\mathbf{D}$  as above is called the *dual spine of  $\mathbf{D}$* . Reversing the construction, we see that each minimal spine  $\mathbf{A}$  gives rise to a minimal disk system  $\mathbf{D}$ , called the *dual disk system of  $\mathbf{A}$* .

Let  $M$  be a connected 3-manifold, and let  $\partial M = \partial_1 M \sqcup \partial_2 M$  be a bipartition of  $\partial M$ . A surface  $S \subset M$  is a *Heegaard surface* for  $(M; \partial_1 M, \partial_2 M)$  if  $S$  separates  $M$  into two compression bodies  $V$  and  $W$  such that  $\partial_+ V = \partial_+ W = S$ ,  $\partial_- V = \partial_1 M$  and  $\partial_- W = \partial_2 M$ ; the decomposition  $M = V \cup_S W$  is called a *Heegaard splitting* of  $(M; \partial_1 M, \partial_2 M)$ . When there is no need to specify the bipartition of  $\partial M$ , we may simply say that  $S$  is a Heegaard surface for  $M$  and  $M = V \cup_S W$  is a Heegaard splitting of  $M$ . The genus of a Heegaard splitting  $M = V \cup_S W$  is defined to be the genus of the Heegaard surface  $S$ . The (*Heegaard*) *genus* of  $M$  is defined to be the minimum of the genera of Heegaard splittings of  $M$ , and it is denoted by  $g(M)$ .

For a knot  $K$  in a connected 3-manifold  $M$ , the *tunnel number*  $t(K)$  is defined to be the minimal number of properly embedded arcs  $\mathbf{A}$  in  $E(K)$  such that  $E(\mathbf{A}, E(K))$  is a handlebody; it is easy to see that

$t(K) = g(E(K)) - 1$ . A knot  $K$  is said to be in a *bridge position with respect to a Heegaard splitting*  $M = V \cup_S W$  if  $K$  meets each compression body in  $\partial$ -parallel arcs [MS91] [Dol92].

An extreme choice of a bipartition of  $\partial M$  is given by  $\partial_1 M = \emptyset$  and  $\partial_2 M = \partial M$ . Following Kobayashi and Nishi [KN94], we call the Heegaard splitting with respect to the bipartition  $\emptyset \sqcup \partial M$  a *tunnel-type Heegaard splitting*. The *tunnel-type genus* of  $M$  is defined to be the minimum of the genera of all tunnel-type Heegaard splittings of  $M$ , and it will be denoted by  $g^t(M)$ . Clearly, every Heegaard splitting of a closed 3-manifold  $M$  is of tunnel-type. When  $\partial M$  consists of a single component, every Heegaard splitting of  $M$  is of tunnel-type up to reordering, since there is a unique bipartition of  $\partial M$  up to reordering.

For a 3-manifold  $M$  that is not connected, a Heegaard splitting of  $M$  is defined to be a union of Heegaard splittings of components of  $M$ . In other words, we have a Heegaard splitting  $M_i = V_i \cup_{F_i} W_i$  for each component  $M_i$ , and a Heegaard splitting of  $M$  is the decomposition  $M = V \cup_F W$  where  $V = \bigcup_i V_i$ ,  $W = \bigcup_i W_i$ , and  $F = \bigcup_i F_i$ . The genus of the splitting is defined to be  $g(F)$ , i.e. the total genus of  $F$ . The tunnel-type splittings and tunnel-type genus are defined analogously for disconnected  $M$  as well.

One of the most fundamental theorems in the modern studies of Heegaard splittings is the Haken–Casson–Gordon Lemma [Hak68] [CG87] and its consequences. Hayashi and Shimokawa [HS01] gave a generalization, which allows the presence of 1-submanifold and employs the notion of compression bodies used in this paper. We will restate a special case below.

**Theorem 6** [HS01, Theorem 1.3]. *Let  $M$  be a connected 3-manifold with non-empty boundary, and let  $L$  be a (possibly empty) link in  $M$ . Let  $M = V \cup_S W$  be a Heegaard splitting of  $M$ , and suppose that, if  $L \neq \emptyset$ ,  $L$  is in a bridge position with respect to  $S$ . If  $D \subset E(L)$  is a compression disk for  $\partial_- W$ , then there exists a compression disk  $D' \subset E(L)$  such that*

- (1)  $S \cap D'$  is an essential simple closed curve on  $S$ ,
- (2)  $D'$  is obtained from  $D$  by 2-surgeries and isotopy in  $E(L)$ ,
- (3) if  $E(L)$  is irreducible, then  $D'$  is isotopic to  $D$  in  $E(L)$ , and
- (4) there exist minimal disk systems of  $V$  and  $W$  disjoint from  $D'$ .

*Remark.* The conclusions (3) and (4) were not explicitly stated in [HS01]. We note that (3) readily follows from the proof of (1) and (2) in [HS01]. Also, the argument in [CG83], where a statement analogous to (4) was given, easily carries over to our setting with minor modification to yield (4).

Following Kobayashi and Nishi, we can now describe how the tunnel-type genus of a 3-manifold  $M$  changes after cutting it along a compression disk for the boundary.

**Proposition 7** (c.f. [KN94, Proposition 6.2]). *Let  $M$  be a 3-manifold with non-empty boundary, and let  $L$  be a (possibly empty) link in  $M$  such that  $E(L)$  is irreducible. Suppose  $D \subset E(L)$  is a compression disk for  $\partial M$ , and let  $M_- := E(D, M)$  be the manifold obtained by cutting  $M$  along  $D$ . Then,*

$$g^t(M_-) = \begin{cases} g^t(M) & \text{if } |M_-| = |M| + 1, \text{ or} \\ g^t(M) - 1 & \text{if } |M_-| = |M|. \end{cases}$$

*Proof.* Since the compression only affects the component of  $M$  that contains  $D$ , we may assume  $M$  is connected as well. Consider a minimal genus tunnel-type Heegaard splitting  $M = V \cup_S W$  where  $V$  is a handlebody. Isotoping  $L$  if necessary, we may assume that  $L$  is in a bridge position with respect to  $S$ . By Theorem 6, we may isotope  $D$  in  $E(L)$  so that it meets  $S$  in a simple closed essential curve on  $S \cap E(L)$ . The rest of the proof is identical to that of [KN94], and we omit it.  $\square$

### 3. IRREDUCIBLE KNOTS AND SIMPLE KNOTS

Throughout this section, we assume that  $M$  is a closed connected 3-manifold. A knot  $K$  in  $M$  is said to be *irreducible* if the exterior  $E(K)$  is irreducible, and it is said to be *simple* if  $E(K)$  is irreducible and atoroidal. Simple knots comprise an important class of knots in the study of 3-manifolds, since Thurston's hyperbolization theorem implies that the complement of a simple knot admits a unique hyperbolic structure. In this section, we record some facts and observations about these knots and their exterior. Most of the material has appeared elsewhere in some form.

Simple knots are known to exist in every closed connected 3-manifold  $M$ ; generalizing the ideas in [Bin58], Myers gave a construction of a simple knot [Mye82] in  $M$ . His construction starts with a special handle decomposition of  $M$  and produces a simple knot by connecting a sufficiently complicated tangle in the 0-handles by arcs in the 1-handles that are parallel to the core arcs. Hass and Thompson [HT89] showed that this construction could produce an infinite number of such knots in  $M$  that are distinct (i.e. pairwise inequivalent).

**Theorem 8** [Mye82, Theorem 6.1], [HT89, Proposition 3]. *Let  $M$  be a closed connected 3-manifold. Then, there exist an infinite number of distinct simple knots in  $M$ .*

*Remark.* Rieck recently gave a quick proof of the existence of irreducible knots [Rie07] that depends on the theory of Heegaard splittings.

Let  $K$  be an irreducible knot in  $M$ . If  $K$  is contained in a connected submanifold  $Y$ , some components of  $\overline{M - Y}$  can be described concretely.

**Lemma 9.** *Let  $M$  be a closed connected 3-manifold and  $Y$  be a connected 3-submanifold of  $M$ . If there exists a knot  $K \subset Y$  which is irreducible in  $M$ , then each  $S^2$ -component of  $\partial Y$  bounds a closed 3-ball outside  $Y$ .*

*Proof.* Let  $S$  be an  $S^2$ -component of  $\partial Y$ . If a knot  $K \subset Y$  is irreducible in  $M$ ,  $S \subset E(K)$  must bound a closed 3-ball in  $E(K)$ . Such a ball must sit outside  $Y$ , since  $K \subset Y$ .  $\square$

If there exists a compression disk  $D$  for  $\partial Y$  such that  $\partial D$  lies on some torus component  $T \subset \partial Y$ , then compression along  $D$  gives a sphere that bounds a ball; it follows that  $T \subset \partial Y$  bounds a component  $Z$  of  $\overline{M - Y}$ . Extending the observations and arguments that have appeared in [Mye82, §9] and [HT89], we obtain the following lemma.

**Lemma 10.** *Let  $M$  be a closed connected 3-manifold and  $Y$  be a connected 3-submanifold of  $M$ . If there exists a knot  $K \subset Y$  which is irreducible in  $M$ , and if there exists a compression disk  $D \subset E(K)$  for  $\partial Y$  such that  $\partial D$  lies on some torus component  $T \subset \partial Y$ , then  $T$  bounds a submanifold  $Z$  outside of  $Y$  such that  $Z$  is homeomorphic to the exterior of some knot in  $S^3$ . Moreover, there exists a re-embedding of  $Y$  into  $M$ , restricting to the identity map on  $K \subset Y$ , such that the image of  $T$  bounds a solid torus outside the image of  $Y$ .*

*Proof.* Let  $S$  denote the 2-sphere obtained by compressing  $T$  along  $D$ .

If  $D \subset \overline{M - Y}$ , compression removes  $N(D, \overline{M - Y})$  from  $\overline{M - Y}$ . By the irreducibility of  $E(K)$ , the 2-sphere  $S$  must bound a 3-ball  $B$  outside  $Y$ . Reversing the compression, we see that  $T$  bounds a solid torus  $Z := B \cup \overline{N(D, \overline{M - Y})}$ .

If  $D \subset Y$ , compression along  $D$  removes  $N(D, E(K) \cap Y)$  from  $Y$ ; here,  $N(D, E(K) \cap Y)$  can be regarded as a 1-handle in  $Y$  with  $D$  as the cocore. By the irreducibility of  $E(K)$ , the 2-sphere  $S$  must bound a closed ball  $B$  such that  $B \cap Y$  is precisely the 1-handle that we removed from  $Y$ . Reversing the compression, we recover  $Y$  by adding this 1-handle back, which simultaneously drills out the regular neighborhood of the core of this 1-handle from  $B$  to yield  $Z$ . By construction,  $Z$  is the exterior of some knot in  $S^3$ .

To prove the last assertion, we only need to modify the last step in the  $D \subset Y$  case above. Instead of recovering  $Y$  by adding the 1-handle  $N(D, E(K) \cap Y)$  back to  $Y - N(D, E(K) \cap Y)$ , we can add a trivial 1-handle inside  $B$  to  $Y - N(D, E(K) \cap Y)$ . We obtain a submanifold  $Y'$  homeomorphic to  $Y$ , and the new torus boundary now bounds a solid torus. It is easy to see that  $Y$  re-embeds as  $Y'$  with the desired properties.  $\square$

Now, suppose that  $K$  is simple in  $M$  so that every  $T^2$ -component of  $\partial Y$  is compressible in  $E(K)$ , except for the  $\partial$ -parallel torus. As it turns out, we can avoid  $\partial$ -parallel torus by imposing a lower bound on the tunnel number of  $K$ . For now, let us illustrate this idea by giving the following lemma, which treats the case where  $\partial Y$  consists of a single  $T^2$ -component. We will give a generalization that allows higher genera and multiple components in Section 5.

**Lemma 11.** *Let  $M$  be a closed connected 3-manifold and  $Y$  be a connected 3-submanifold of  $M$  with  $\partial Y \cong T^2$ , and let  $Z := \overline{M - Y}$ . If there exists a knot  $K \subset Y$  which is simple in  $M$  with  $t(K) \geq g(Z)$ , then  $Z$  is homeomorphic to the exterior of some knot in  $S^3$ . Moreover, there exists a re-embedding of  $Y$  into  $M$ , restricting to the identity map on  $K \subset Y$ , such that the image of  $\partial Y$  bounds a solid torus outside the image of  $Y$ .*

*Proof.* If  $\partial Y$  is  $\partial$ -parallel in  $E(K)$ ,  $Y$  must be a solid torus with  $K$  as its core. Then, it is easy to see that  $g(Z) = t(K) + 1 > t(K)$ , which contradicts our assumption. Thus, by the definition of simple knots,  $\partial Y \subset E(K)$  must be a compressible torus in  $E(K)$ . The conclusion follows from Lemma 10.  $\square$

#### 4. AMALGAMATED HEEGAARD GENUS

When a connected 3-manifold  $M$  is separated by a closed surface  $F$  into two submanifolds  $Y_1$  and  $Y_2$ , there is a natural construction of a Heegaard splitting of  $M$  from that of  $Y_1$  and  $Y_2$ , say  $Y_1 = V_1 \cup_{S_1} W_1$  and  $Y_2 = V_2 \cup_{S_2} W_2$  with  $F = \partial_- W_1 \cap \partial_- W_2$ . Loosely speaking, one combines  $V_1$  and  $W_2$  on one side,  $W_1$  and  $V_2$  on the other, by collapsing the collar neighborhood of  $F$ . This construction, called *amalgamation*, was first defined by Schultens in [Sch93].

Let us describe the amalgamation process briefly for the case where we start with tunnel-type Heegaard splittings  $Y_i = V_i \cup_{S_i} W_i$  for each  $i$ , assuming that  $V_i$  is a union of handlebodies and  $W_i$  is a union of compression bodies. Writing  $\partial_i M := \partial Y_i - F$ , we have  $\partial_- W_i = \partial Y_i = F \cup \partial_i M$  and  $\partial M = \partial_1 M \cup \partial_2 M$ . By definition,  $W_i$  is obtained by

attaching 1-handles to  $\overline{N}(\partial Y_i, Y_i) \cong \partial_- W_i \times I$ . Some of these 1-handles have one or both ends attached to  $\overline{N}(F, Y_i) \cong F \times I$ . One can extend these 1-handles vertically through the product region  $\overline{N}(F, Y_i)$  so that they are attached to  $F$ , perturbing if necessary so that the attaching disks for handles from opposite sides are disjoint on  $F$ . Now, take the union of the boundaries of all 1-handles (some of which are extended) together with  $F$  and  $\partial \overline{N}(\partial M_i, Y_i) - \partial M_i$ , and remove the interior of attaching disks. One can check that the resulting surface is indeed a Heegaard surface for  $(M; \partial_1 M, \partial_2 M)$ . If each component of  $F$  is separating, the genus of this Heegaard surface obtained by amalgamation is  $g(S_1) + g(S_2) - g(F)$ .

A more general definition of the amalgamation does not require the splittings of  $Y_i$  to be of tunnel-type, and the components of  $\partial Y_i - F$  are allowed to be in either  $\partial_- V_i$  or  $\partial_- W_i$ . However, if  $M$  is closed so that  $M = Y_1 \cup_F Y_2$  with  $\partial Y_1 = F = \partial Y_2$ , the Heegaard splitting of each  $Y_i$  must always be a tunnel-type splitting.

For a closed 3-manifold  $M$  and a separating surface  $F$ , we define the *amalgamated Heegaard genus of  $M$  with respect to  $F$* , denoted by  $g(M; F)$ , to be the smallest possible genus of a Heegaard splitting of  $M$  obtained by amalgamation along  $F$ . If each component of  $F$  is separating, it is easy to see that the amalgamated Heegaard genus  $g(M; F)$  is minimized if and only if we amalgamate the minimal genus tunnel-type splittings of  $Y_1$  and  $Y_2$ . Thus, we have

$$g(M; F) = g^t(Y_1) + g^t(Y_2) - g(F),$$

when each component of  $F$  is separating.

**Lemma 12.** *Let  $M$  be a closed connected 3-manifold and  $F \subset M$  be a closed surface that separates  $M$  into  $Y$  and  $Z$  such that  $Y$  is connected. If  $F$  consists of components  $F_1, \dots, F_n$ , such that each  $F_i$  separates  $M$ , then we have  $g(F_i; M) \leq g(F; M)$  for each  $i$ .*

*Proof.* Since  $Y$  is connected and each  $F_i$  is separating, it follows that each  $F_i$  bounds a component of  $Z$ . It follows that  $Z$  consists of  $n$  components, and we can denote them as  $Z_1, \dots, Z_n$  such that  $\partial Z_i = F_i$  for each  $i$ .

For each  $i$ , let us write  $Z_i^* := Z - Z_i$  and  $F_i^* = F - F_i$  so that  $\partial Z_i^* = F_i^*$ . Also let  $Y_i = \overline{M - Z_i} = Y \cup_{F_i^*} Z_i^*$ , so that  $\partial Y_i = F_i$  and  $M = Z_i \cup_{F_i} Y_i$ . Note that the amalgamation of tunnel-type Heegaard splittings of  $Y$  and  $Z_i^*$  along  $F_i^*$  produces a Heegaard splitting of  $Y_i$ . Starting with the minimal genus tunnel-type splittings of  $Y$  and  $Z_i^*$ ,

we have

$$g^t(Y_i) \leq g^t(Y) + g^t(Z_i^*) - g(F_i^*) = g^t(Y) + \sum_{j \neq i} g^t(Z_j) - \sum_{j \neq i} g(F_j).$$

Thus, we obtain

$$\begin{aligned} g(M; F_i) &= g^t(Y_i) + g^t(Z_i) - g^t(F_i) \\ &\leq \left( g^t(Y) + \sum_{j \neq i} g^t(Z_j) - \sum_{j \neq i} g(F_j) \right) + g^t(Z_i) - g(F_i) \\ &= g^t(Y) + \sum_j g^t(Z_j) - \sum_j g(F_j) \\ &= g^t(Y) + g^t(Z) - g(F) \\ &= g(M; F). \end{aligned}$$

□

As Rieck pointed out to us [Rie] (see [KR06, §2]), if some components of  $F$  are not separating, the genus of the Heegaard surface obtained by amalgamating the Heegaard splittings  $Y_i = V_i \cup_{S_i} W_i$  is not  $g(S_1) + g(S_2) - g(F)$ . For example, suppose  $M = Y_1 \cup_F Y_2$  where each  $Y_i$  is connected and  $F$  consists of  $m > 1$  components, so that every component of  $F$  is non-separating; in this case, the genus of the Heegaard surface obtained by amalgamation is  $g(S_1) + g(S_2) - g(F) + (m - 1)$ , and hence the amalgamated Heegaard genus is given by the formula

$$g(M; F) = g^t(Y_1) + g^t(Y_2) - g(F) + (m - 1).$$

Such situations do not occur when we utilize the notion of amalgamated Heegaard genus in the proof of our main theorem. Nonetheless, we consider these cases as well in the following lemma, which may be of independent interests.

**Lemma 13.** *Let  $M$  be a closed connected 3-manifold and  $F \subset M$  be a closed connected surface that separates  $M$ . Suppose  $L \subset M - F$  is a (possibly empty) link in  $M$  such that  $E(L)$  is irreducible. If  $F$  is compressible in  $E(L)$  and  $F_\circ$  is the surface obtained by compressing  $F$  along a compression disk in  $E(L)$ , then  $F_\circ$  also separates  $M$  and  $g(F_\circ; M) \leq g(F; M)$ ; moreover, the equality  $g(F_\circ; M) = g(F; M)$  holds if and only if  $g(Z_+) = g(Z)$ .*

*Proof.* Suppose  $M = Y \cup_F Z$ , i.e.  $F$  separates  $M$  into submanifolds  $Y$  and  $Z$ . Clearly, the surface  $F_\circ$  obtained by compressing the separating surface  $F$  is also separating. Without loss of generality, assume that there is a compression disk  $D \subset E(L) \cap Y$  for  $F$ . Let  $Y_- := Y - N(D, E(L) \cap Y)$  and  $Z_+ := \overline{M - Y_-} = Z \cup \overline{N(D, E(L) \cap Y)}$ , so

that we can set  $F_\circ := \partial Y_- = \partial Z_+$ . Since any Heegaard surface of  $Z$  can be regarded as a tunnel-type Heegaard surface of the connected submanifold  $Z_+$ , we have  $g^t(Z_+) \leq g^t(Z)$ .

Suppose  $\partial D$  lies on  $F$  as a separating loop, so that  $g(F_\circ) = g(F)$ . If  $D$  lies in  $Y$  as a separating disk, we have  $g^t(Y_-) = g^t(Y)$  by Proposition 7. Thus, we obtain

$$\begin{aligned} g(M; F_\circ) &= g^t(Y_-) + g^t(Z_+) - g(F_\circ) \\ &= g^t(Y) + g^t(Z_+) - g(F) \\ &\leq g^t(Y) + g^t(Z) - g(F) = g(M; F). \end{aligned}$$

If  $D$  lies in  $Y$  as a non-separating disk, we have  $g^t(Y_-) = g^t(Y) - 1$  by Proposition 7. In this case, note that  $F_\circ$  consists of two components, and each of these components is non-separating. Using the formula in the earlier remark, we obtain

$$\begin{aligned} g(M; F_\circ) &= g^t(Y_-) + g^t(Z_+) - g(F_\circ) + 1 \\ &= (g^t(Y) - 1) + g^t(Z_+) - g(F) + 1 \\ &= g^t(Y) + g^t(Z_+) - g(F) \\ &\leq g^t(Y) + g^t(Z) - g(F) = g(M; F). \end{aligned}$$

Now, suppose  $\partial D$  lies on  $F$  as a non-separating loop, so that  $g(F_\circ) = g(F) - 1$ . In this case,  $D$  lies in  $Y$  as a non-separating disk, and we have  $g^t(Y_-) = g^t(Y) - 1$  by Proposition 7. Thus, we obtain

$$\begin{aligned} g(M; F_\circ) &= g^t(Y_-) + g^t(Z_+) - g(F_\circ) \\ &= (g^t(Y) - 1) + g^t(Z_+) - (g(F) - 1) \\ &= g^t(Y) + g^t(Z_+) - g(F) \\ &\leq g^t(Y) + g^t(Z) - g(F) = g(M; F). \end{aligned}$$

In all cases, we have established

$$\begin{aligned} g(M; F_\circ) &= g^t(Y) + g^t(Z_+) - g(F) \\ &\leq g^t(Y) + g^t(Z) - g(F) = g(M; F), \end{aligned}$$

and the lemma follows.  $\square$

## 5. CHARACTERISTIC KNOTS

Let  $M$  be a closed connected 3-manifold. Generalizing the notions of irreducible knots and simple knots in a natural way, one can study a class of knots  $K$  whose exterior  $E(K)$  contains no closed connected orientable incompressible surface of genus less than or equal to  $g$  except for the  $\partial$ -parallel torus. In the terminology of Kobayashi and Nishi [KN94], such knots are said to be  *$g$ -characteristic in  $M$* . Irreducible knots and simple knots in a closed connected 3-manifold are precisely 0-characteristic knots and 1-characteristic knots respectively.

Generalizing the Myers' construction, Kobayashi and Nishi [KN94] established the existence of  $g$ -characteristic knots for all  $g \geq 0$ . They further showed the following, which we restate here in terms of the tunnel number of knots.

**Theorem 14** [KN94, Theorem 5.1]. *Let  $M$  be a closed connected 3-manifold. Then, for any integers  $g \geq 0$  and  $t \geq 0$ , there exist an infinite number of  $g$ -characteristic knots  $K \subset M$  with  $t(K) \geq t$ .*

*Remark.* If a knot  $K$  is  $g$ -characteristic for all  $g \geq 0$ , it is often called a *small* knot in the literature. The existence of small knots in a closed 3-manifold is not known in general. It is conjectured that every non-Haken manifold contains a hyperbolic small knot. The existence of small knots for some special cases can be found in [Lop93], [Mat04], and [QW04].

From the definition of  $g$ -characteristic knots, we can deduce some useful properties of the exterior of such knots. In particular, closed orientable surfaces of small genera in the exterior of  $g$ -characteristic knots share some properties with closed orientable surfaces in irreducible non-Haken 3-manifolds.

**Lemma 15.** *Fix an integer  $g \geq 0$ , and let  $M$  be a closed connected 3-manifold. If there exists a knot  $K \subset M$  which is  $g$ -characteristic in  $M$ , then every connected orientable surface  $F \subset E(K)$  with  $g(F) \leq g$  must separate  $M$ .*

*Proof.* Suppose there is a non-separating connected orientable surface  $F \subset E(K)$  with  $g(F) \leq g$ . Since  $K$  is  $g$ -characteristic,  $F$  must either be compressible or  $\partial$ -parallel in  $E(K)$ . Since  $F$  is clearly separating in the latter case, we may assume that  $F$  is compressible in  $E(K)$ . Compressing  $F$  along a compression disk  $D \subset E(K)$  and keeping a non-separating component, we obtain a non-separating connected orientable surface with strictly smaller genus. The process must eventually end with an incompressible surface  $F'$  with  $g(F') \leq g$ , which contradicts the assumption.  $\square$

Let  $Y$  be a connected 3-submanifold of  $M$ , and suppose that  $Y$  contains a knot  $K$  which is  $g$ -characteristic in  $M$ . In this situation, Lemma 15 implies that a component  $F \subset \partial Y$  with  $g(F) \leq g$  must bound a submanifold  $Z$  outside  $Y$  such that  $\partial Z = F$ .

The next proposition generalizes the re-embeddability statement of Lemma 11 to the cases where  $\partial Y$  is a connected surface of higher genus. The key assumption is that the tunnel number of the characteristic knot is bounded by the amalgamated genus.

**Proposition 16.** *Let  $M$  be a closed connected 3-manifold and  $Y$  be a connected 3-submanifold of  $M$  with non-empty connected boundary  $F := \partial Y$ . If there exists a knot  $K \subset Y$  which is  $g(F)$ -characteristic in  $M$  with  $t(K) \geq g(M; F)$ , then there exists a re-embedding of  $Y$  into  $M$ , restricting to the identity map on  $K \subset Y$ , such that the image is a Fox submanifold of  $M$ .*

*Proof.* If  $g(F) = 0$ , it follows from Lemma 9 that  $F$  bounds a closed 3-ball outside  $Y$ . In other words,  $Y$  is already a Fox submanifold, and no re-embedding is necessary. For  $g(F) \geq 1$ , we prove the proposition by induction on  $g(F)$ . Let  $Z := \overline{M - Y}$  so that  $F = \partial Y = \partial Z$ . For the base case, suppose  $g(F) = 1$ . Then,  $t(K) \geq g(M; F) = g^t(Y) + g^t(Z) - g(F) \geq 1 + g(Z) - 1 = g(Z)$ , and the statement follows from Lemma 11.

Now, for the induction, let  $g > 1$  and suppose that the statement holds when  $1 \leq g(F) < g$ ; we aim to show that the statement also holds when  $g(F) = g$ . By assumption, there exists a knot  $K \subset Y$  which is  $g(F)$ -characteristic in  $M$  with  $t(K) \geq g(M; F)$ . Since  $g(F) = g > 1$ ,  $F$  must be compressible in  $E(K)$ . We consider the following four cases separately.

*Case 1.* There is a compression disk  $D \subset Z$  for  $F$  such that  $\partial D$  does not separate  $F$ .

Let  $Y_+$  be the manifold obtained from  $Y$  by adding a 2-handle  $Q = \overline{N(D, Z)}$ . Let  $Z_- := \overline{M - Y_+}$  and  $F_\circ := \partial Y_+ = \partial Z_-$ . Since  $F_\circ$  is the surface obtained by compressing  $F$  along  $D$ ,  $F_\circ$  is a connected surface with  $g(F_\circ) = g(F) - 1 = g - 1 \geq 1$ .

We already know that the knot  $K \subset Y_+$  is  $(g-1)$ -characteristic in  $M$ . Also, by Lemma 13,  $g(M; F_\circ) \leq g(M; F) \leq t(K)$ . Thus, invoking the induction hypothesis, there exists a re-embedding of  $Y_+$ , which restricts to the identity map on  $K \subset Y_+$ , with the image  $Y'_+ \subset M$  so that  $Z'_- := \overline{M - Y'_+}$  is a handlebody. Note that we have a decomposition  $Y'_+ = Y' \cup Q'$  where  $Y' \supset K$  and  $Q'$  are the images of  $Y \supset K$  and  $Q$  respectively. Moreover,  $\overline{M - Y'} = Z'_- \cup Q'$  is a handlebody, since  $Q'$  is added to the handlebody  $Z'_-$  as a 1-handle. One can now easily obtain a re-embedding of  $Y$  onto  $Y' \cong Y$  with the desired properties.

*Case 2.* There is a compression disk  $D \subset E(K) \cap Y$  for  $F$  such that  $\partial D$  does not separate  $F$ .

Let  $Z_+$  be the manifold obtained from  $Z$  by adding a 2-handle  $Q = \overline{N(D, E(K) \cap Y)}$ . Let  $Y_- := \overline{M - Z_+}$  and  $F_\circ := \partial Y_- = \partial Z_+$ . By the argument analogous to Case 1 above, there exists a re-embedding of

$Y_-$ , which restricts to the identity map on  $K \subset Y_-$ , with the image  $Y'_- \subset M$  so that  $Z'_+ := \overline{M - Y'_-}$  is a handlebody.

Now, let  $\alpha$  be a properly embedded  $\partial$ -parallel arc in  $Z'_+$ . Then,  $Y' := Y'_- \cup \overline{N}(\alpha, Z'_+) \supset K$  is a homeomorphic copy of  $Y$ , and  $\overline{M - Y'} = Z'_+ - N(\alpha, Z'_+)$  is a handlebody. One can now easily obtain a re-embedding of  $Y$  onto  $Y' \cong Y$  with the desired properties.

*Case 3.* There is a compression disk  $D \subset Z$  for  $F$  such that  $\partial D$  separates  $F$ .

Let  $Y_+$  be the manifold obtained by adding a 2-handle  $Q = \overline{N}(D, Z)$  along the neighborhood of a separating essential loop  $\partial D$  on  $F$ . Let  $Z_- := \overline{M - Y_+}$  and  $F_\circ := \partial Y_+ = \partial Z_-$ . Since  $F_\circ$  is the surface obtained by compressing  $\partial Y$  along  $D$ ,  $F_\circ$  is a two-component surface, say  $F_\circ = F_{\circ,1} \sqcup F_{\circ,2}$ , with  $g(F_{\circ,1}) + g(F_{\circ,2}) = g(F)$  and  $g(F) > g(F_{\circ,i}) \geq 1$ . Note that, by Lemma 15, each  $F_{\circ,i}$  is a separating surface. Thus, one sees that  $D$  must have been a separating disk for  $Z$ , and that  $Z_-$  consists of two components. Write these components as  $Z_{-,1}$  and  $Z_{-,2}$  so that  $\partial Z_{-,i} = F_{\circ,i}$ .

Let  $Y_{+,1} := Y_+ \cup_{F_{\circ,2}} Z_{-,2}$ , so that  $\partial Y_{+,1} = F_{\circ,1} = \partial Z_{-,1}$ . We already know that the knot  $K \subset Y_+ \subset Y_{+,1}$  is  $g(F_{\circ,1})$ -characteristic. Also, by Lemma 12 and Lemma 13,  $g(M; F_{\circ,1}) \leq g(M; F_\circ) \leq g(M; F) \leq t(K)$ . Thus, invoking the induction hypothesis, there exists a re-embedding of  $Y_{+,1}$ , restricting to the identity map on  $K \subset Y_{+,1}$ , with the image  $Y'_{+,1} \subset M$  so that  $Z'_{-,1} := \overline{M - Y'_{+,1}}$  is a handlebody. Let us write  $F'_{\circ,1}$  for the image of  $F_{\circ,1}$  so that  $\partial Y'_{+,1} = F'_{\circ,1} = \partial Z'_{-,1}$ . Note that we have a decomposition  $Y'_{+,1} = Y'_+ \cup_{F'_{\circ,2}} Z'_{-,2}$ , where  $Y'_+$ ,  $F'_{\circ,2}$ , and  $Z'_{-,2}$  are the images of  $Y_+$ ,  $F_{\circ,2}$ , and  $Z_{-,2}$  respectively. We also note that  $K \subset Y'_+$  since  $K \subset Y_+$  before the re-embedding.

Let  $Y'_{+,2} := Y'_+ \cup_{F'_{\circ,1}} Z'_{-,1}$  so that  $\partial Y'_{+,2} = F'_{\circ,2} = \partial Z'_{-,2}$ . In order to repeat the argument in the last paragraph, let us first verify the necessary inequality to invoke the induction hypothesis. Writing  $F'_\circ = F'_{\circ,1} \sqcup F'_{\circ,2}$ , we have  $F'_\circ \cong F_\circ$  and thus  $g(F'_\circ) = g(F_\circ)$ . Also, since  $Y'_+ \cong Y_+$  and  $Z'_{-,2} \cong Z_{-,2}$ , we have  $g^t(Y'_+) = g^t(Y_+)$  and  $g^t(Z'_{-,2}) = g^t(Z_{-,2})$ . On the other hand, since  $Z'_{-,1}$  is a handlebody,  $g^t(Z'_{-,1}) = g(F'_{\circ,1}) = g(F_{\circ,1}) \leq g^t(Z_{-,1})$ . Thus, we have

$$\begin{aligned}
g(M; F'_\circ) &= g^t(Y'_+) + g^t(Z'_-) - g(F'_\circ) \\
&= g^t(Y'_+) + g^t(Z'_{-,1}) + g^t(Z'_{-,2}) - g(F'_\circ) \\
&\leq g^t(Y_+) + g^t(Z_{-,1}) + g^t(Z_{-,2}) - g(F_\circ) \\
&= g^t(Y_+) + g^t(Z_-) - g(F_\circ) \\
&= g(M; F_\circ).
\end{aligned}$$

Together with Lemma 12 and Lemma 13, we now obtain

$$g(M; F'_{\circ,2}) \leq g(M; F'_\circ) \leq g(M; F_\circ) \leq g(M; F) \leq t(K),$$

which is the desired inequality for the induction.

Of course,  $K \subset Y'_+ \subset Y'_{+,2}$  is  $g(F'_{\circ,2})$ -characteristic, since  $K$  is  $g(F)$ -characteristic by assumption and  $g(F'_{\circ,2}) = g(F_{\circ,2}) \leq g(F)$ . Thus, invoking the induction hypothesis, there exists a re-embedding of  $Y'_{+,2}$ , restricting to the identity map on  $K \subset Y'_{+,2}$ , with the image  $Y''_{+,2}$  so that  $Z''_{-,2} := \overline{M - Y''_{+,2}}$  is a handlebody. Note that we have a decomposition  $Y''_{+,2} = Y''_+ \cup_{F''_{\circ,1}} Z''_{-,1}$ , where  $Y''_+$ ,  $F''_{\circ,1}$ , and  $Z''_{-,1}$  are the images of  $Y'_+$ ,  $F'_{\circ,1}$ , and  $Z'_{-,1}$  respectively. In particular, we have a decomposition  $M = Y''_+ \cup Z''_{-,1} \cup Z''_{-,2}$ , where each  $Z''_{-,i}$  is a handlebody. We also have a decomposition  $Y''_+ = Y'' \cup Q''$  where  $Y'' \supset K$  and  $Q''$  are images of  $Y \supset K$  and  $Q$  respectively under the composition of the two re-embeddings. The cocore of the 2-handle  $Q''$  is an arc connecting  $F''_{\circ,1}$  and  $F''_{\circ,2}$ ; thus,  $Q''$  can be regarded as a 1-handle that connects handlebodies  $Z''_{-,1}$  and  $Z''_{-,2}$ . It follows that  $\overline{M - Y''} = Z''_{-,1} \cup Q'' \cup Z''_{-,2}$  is a handlebody. One can now easily obtain a re-embedding of  $Y$  onto  $Y'' \cong Y$  with the desired properties.

*Case 4.* For every compression disk  $D \subset E(K)$  for  $F$ ,  $D$  lies in  $E(K) \cap Y$  and  $\partial D$  separates  $F$ .

We compress  $F$  along a compression disk in  $E(K) \cap Y$ , and continue compressing further along one compression disk at a time if the surface obtained from  $F$  is compressible into the remnant of  $E(K) \cap Y$ . Isotoping the compression disk in each step if necessary, we may assume that its boundary lies in the remnant of  $F$ . Then, it follows that each disk in this process is indeed a compression disk for  $F$  into  $E(K) \cap Y$ . The sequence of such compressions must stop, since  $F$  admits at most  $g(F) - 1$  2-surgeries along non-separating loops. Let us write  $D_1, \dots, D_n$  for these compression disks, and let  $\mathbf{D} := D_1 \sqcup \dots \sqcup D_n \subset E(K) \cap Y$ .

Take  $\mathbf{Q} := \overline{N(\mathbf{D}, E(K) \cap Y)}$ , and let  $Z_+ := Z \cup \mathbf{Q}$ ,  $Y_- := \overline{M - Z_+}$ , and  $F_\circ := \partial Y_- = \partial Z_+$ . By construction,  $F_\circ$  is incompressible into  $E(K) \cap Y$ . It also follows that  $F_\circ$  consists of  $n + 1$  components, say  $F_\circ = F_{\circ,1} \sqcup \dots \sqcup F_{\circ,n+1}$  with  $\sum_{i=1}^{n+1} g(F_{\circ,i}) = g(F)$  and  $g(F) > g(F_{\circ,i}) \geq 1$ . Note that, by Lemma 15, each  $F_{\circ,i}$  is a separating surface. Thus, one sees that each disk in  $\mathbf{D}$  must have been a separating disk for  $Y$ , and that  $Y_-$  consists of  $n + 1$  components. Let  $Y_- = Y_{-,1} \sqcup \dots \sqcup Y_{-,n+1}$  so that  $\partial Y_{-,i} = F_{\circ,i}$ . We remark that  $Y$  is the boundary connect sum of components of  $Y_-$ .

Let  $Z = V \cup_S W$  be a minimal genus (tunnel-type) Heegaard splitting with  $\partial_- V = \emptyset$ , i.e.  $V$  is a handlebody and  $\partial_- W = \partial Z$ . Then,  $W_+ := W \cup \mathbf{Q}$  is a compression body with  $\partial_- W_+ = \partial Z_+$  and  $\partial_+ W_+ = \partial_+ W = S = \partial V$ . It follows that  $Z_+ = V \cup_S W_+$  is a tunnel-type Heegaard splitting of  $Z_+$ .

*Claim.* There exists a component of  $F_\circ$ , which is compressible into  $Z_+$ .

*Proof of Claim.* Suppose that no component of  $F_\circ$  is compressible into  $Z_+$ . Then, since  $K$  is  $g(F)$ -characteristic,  $F_\circ$  consists of  $\partial$ -parallel tori in  $E(K)$ . One can then deduce that  $F_\circ = \partial Z_+$  must consist of precisely two tori and  $Z_+ \cong T^2 \times I$ . Let  $Y_- = Y_{-,1} \sqcup Y_{-,2}$  with  $K \subset Y_{-,1}$ . Note that  $Y_{-,1} \cup Z_+$  is a solid tori with the knot  $K$  as its core. Thus, we have  $Y_{-,2} \cong E(K)$ , and

$$t(K) = g^t(Y_{-,2}) - 1 < g^t(Y_{-,2}).$$

On the other hand, we have

$$\begin{aligned} t(K) &\geq g(M; F) \\ &\geq g(M; F_\circ) \\ &= g^t(Y_{-,1}) + g^t(Y_{-,2}) + g^t(Z_+) - g(F_{\circ,1}) - g(F_{\circ,2}) \\ &= 1 + g^t(Y_{-,2}) + 2 - 1 - 1 \\ &= g^t(Y_{-,2}) + 1 \\ &> g^t(Y_{-,2}), \end{aligned}$$

where the first inequality holds by assumption, and the second inequality follows from Lemma 13. The above inequalities give the desired contradiction, and the claim follows.  $\square$

Let  $D$  be a compression disk of  $F_\circ$  into  $Z_+$ , as in the above claim. By the version of Haken's Lemma due to Casson and Gordon (c.f. Theorem 6), we can isotope  $D$  so that  $D \cap S$  is a simple closed essential curve in  $S$  and find a minimal disk system  $\mathbf{E}$  for  $W_+$  such that  $\mathbf{E} \cap D = \emptyset$ . Note that the dual spine  $\mathbf{A}$  of  $\mathbf{E}$  is a minimal spine consisting of properly embedded arcs in  $W_+$ , and it is also disjoint from  $D$ .

One can choose a subcollection  $\mathbf{A}'$  of the spine  $\mathbf{A}$ , such that (i)  $\mathbf{A}' \cup \partial_- W_+$  is connected, and (ii) the number of arcs in  $\mathbf{A}'$  is minimal among subcollections that satisfy (i). Since  $\partial_- W_+$  consists of  $n + 1$  components, one can see that  $\mathbf{A}'$  consists of precisely  $n$  arcs. Now, take  $\mathbf{Q}' := \overline{N}(\mathbf{A}, W_+ - D)$ , and let  $Y' := Y_- \cup \mathbf{Q}'$ . Also, let  $Z' := \overline{M} - Y'$  and  $F' := \partial Y' = \partial Z'$ . The compression disk  $D$  can now be regarded as a compression disk for  $F'$  into  $Z'$ .

Note that  $Y'$  is obtained from  $Y_-$  by re-connecting the components of  $Y_-$  by 1-handles  $\mathbf{Q}'$  along the arcs  $\mathbf{A}'$ . It follows from the choice

of  $\mathbf{A}'$  that  $Y'$  is indeed the boundary connect sum of components of  $Y_-$ . Thus,  $Y'$  is homeomorphic to  $Y$ , and we have  $g^t(Y') = g^t(Y)$  and  $g(F') = g(F)$ . Moreover, one can easily construct the re-embedding of  $Y$  with image  $Y'$  so that it restricts to the identity map on  $K$ .

We also observe that  $Z'$  can be decomposed as  $Z' = V \cup_{\Sigma} W'$ , where  $W' := \overline{W_+ - \mathbf{Q}'}$ . Since  $W'$  is obtained from the compression body  $W_+$  by drilling out  $\mathbf{Q}'$  along the subcollection of spinal arcs, we see that  $W'$  is again a compression body. The decomposition  $Z' = V \cup_{\Sigma} W'$  is indeed a Heegaard splitting for  $Z'$ , and we have  $g^t(Z') \leq g(\Sigma) = g^t(Z)$ . Now, combined with the equalities from the last paragraph, we obtain

$$\begin{aligned} g(M; F') &= g^t(Y') + g^t(Z') - g(F') \\ &\leq g^t(Y) + g^t(Z) - g(F) = g(M; F) \leq t(K). \end{aligned}$$

Since  $F' = \partial Y'$  compresses into  $Z' = \overline{M - Y'}$ , we can now apply Case 1 or Case 3 to  $Y'$  to obtain the desired re-embedding of  $Y$ .

These four cases together complete the proof of Proposition 16.  $\square$

In Proposition 16, the submanifold  $Y$  has a single boundary component. We now give the analogous re-embedding statement for the case where  $Y$  has a multiple boundary components.

**Proposition 17.** *Let  $M$  be a closed connected 3-manifold and  $Y$  be a connected 3-submanifold of  $M$  with non-empty boundary  $F := \partial Y$ . If there exists a knot  $K \subset Y$  which is  $g_{\max}(F)$ -characteristic in  $M$  with  $t(K) \geq g(M; F)$ , then there exists a re-embedding of  $Y$  into  $M$ , restricting to the identity map on  $K \subset Y$ , such that the image is a Fox submanifold of  $M$ .*

*Proof.* We prove the statement by induction on the number of components  $|F|$  of  $F$ . When  $|F| = 1$ , i.e.  $F$  is connected, the statement coincides with Proposition 16. For the induction, let  $m > 1$  and suppose that the statement holds when  $1 \leq |F| < m$ ; we aim to show that the statement also holds for the case  $|F| = m$ .

Let us write  $F_i$  for each component of  $F$  so that  $F := F_1 \sqcup \cdots \sqcup F_m$ , and let  $Z := \overline{M - Y}$  as before. Since  $K \subset Y$  is  $g_{\max}(F)$ -characteristic with  $g_{\max}(F) \geq g(F_i)$ , we know from Lemma 15 that each  $F_i$  must be a separating surface. It follows that each  $F_i$  bounds a component of  $Z$  outside  $Y$ . Let  $Z = Z_1 \sqcup \cdots \sqcup Z_m$  with  $\partial Z_i = F_i$ .

Take  $Y_1 := Y \cup Z_2 \cup \cdots \cup Z_m$ , so that  $\partial Y_1 = \partial Z_1 = F_1$ . By the argument analogous to Case 3 in the proof of Proposition 16, we can invoke induction hypothesis to obtain a re-embedding of  $Y_1$  onto  $Y'_1$ , restricting to the identity on  $K$ , so that the image  $F'_1$  of  $F_1$  bounds a handlebody  $Z'_1$  outside  $Y'_1$ . Note that  $Y'_1$  decomposes as  $Y'_1 := Y' \cup$

$Z'_2 \cup \cdots \cup Z'_m$ , where  $Y', Z'_2, \cdots, Z'_m$  are the images of  $Y, Z_2, \cdots, Z_m$  respectively. For  $i = 2, \cdots, m$ , the image  $F'_i$  of  $F_i$  bounds  $Z'_i$  outside  $Y'$ .

Now, take  $Y'_0 := Y' \cup_{F'_1} Z'_1$ ,  $\partial Y'_0 = \bigsqcup_{i=2}^m F'_i$ . Again by the argument analogous to Case 3 in the the proof of Proposition 16, one can invoke the induction hypothesis to obtain a re-embedding of  $Y'_0$  onto  $Y''_0$ , restricting to the identity on  $K$ , so that the image  $F''_i$  of  $F'_i$  bounds a handlebody  $Z''_i$  for  $i = 2, \cdots, m$ . Note that  $Y''_0$  decomposes as  $Y''_0 = Y'' \cup Z''_1$ , where  $Y''$  and  $Z''_1$  are the images of  $Y'$  and  $Z'_1$  respectively. The image  $F''_1$  of  $F'_1$  also bounds the handlebody  $Z''_1$ .

Thus, composing the two re-embeddings, we have a re-embedding of  $Y$  onto  $Y''$ , restricting to the identity on  $K$ , so that each component  $F''_i$  of  $F'' = \partial Y''$  bounds a handlebody.  $\square$

## 6. RE-EMBEDDING

We can now give our main theorem that relates the notions of Fox submanifolds and Bing Submanifolds of a closed connected 3-manifold. Most of the work is already done in the last section, and the theorem is an immediate consequence of Proposition 17.

Note that every Fox submanifold is already a Bing submanifold although a Bing submanifold is usually not a Fox submanifold. Thus, Theorem 3 can be restated as follows:

**Theorem 18.** *Let  $M$  be a closed connected 3-manifold. Then, every Bing submanifold of  $M$  can be re-embedded into  $M$  as a Fox submanifold of  $M$ .*

*Proof.* Let  $Y$  be a Bing submanifold of a closed connected 3-manifold  $M$ . By Theorem 14, there exists a  $g_{\max}(\partial Y)$ -characteristic knot  $K \subset M$  with  $t(K) \geq g(M; \partial Y)$ . Since  $Y$  is a Bing submanifold of  $M$ , we may isotope  $K$  into  $Y$ ; thus, we may as well assume  $K \subset Y$ . By Proposition 17, we can re-embed  $Y$  as a Fox submanifold  $Y' \cong Y$ .  $\square$

It is worthwhile to record the following examples, which shows that some of the theorems mentioned in Section 1 are essentially special cases of Theorem 18.

**Example.** Suppose  $M \cong S^3$ . It is easy to see that every submanifold of  $S^3$  is a Bing submanifold. Thus, our Theorem 18 specializes to the original Fox re-embedding theorem [Fox48] (cited as Theorem 1 in Section 1).

**Example.** Suppose  $Y \cong B^3$ . Since every embedding of a 3-ball into a connected 3-manifold is isotopic to each other, Theorem 18 says that the closure of the complement of  $Y$  is a single 3-ball as well. In other

words,  $M \cong S^3$ . This is essentially Bing's characterization of  $S^3$  [Bin58] (cited as Theorem 2), since the other direction of Bing's theorem is trivial.

**Example.** Suppose  $Y$  is a genus  $g$  handlebody. Theorem 18 says that there is another genus  $g$  handlebody, say  $Y'$  such that the closure of the complement  $Y'$  is a handlebody of genus  $g$  as well. In other words,  $M = Y' \cup \overline{M - Y'}$  is a Heegaard splitting of  $M$ . This is essentially the main results of [HT89] and [KN94], since the other direction of these theorems is again trivial.

The last two examples together provide the characterization of closed 3-manifolds that admits a genus  $g$  splittings for each  $g \geq 0$ . As an immediate consequence of Theorem 18, we can also give the following alternative characterization of such manifolds.

**Corollary 4.** *A closed connected 3-manifold  $M$  admits a genus  $g$  Heegaard splitting if and only if there exists a closed orientable surface  $F$  in  $M$  with genus  $g$  such that every knot in  $M$  can be isotoped to lie within a regular neighborhood of  $F$ .*

*Proof.* The statement follows immediately if we apply Theorem 18 to the neighborhood  $Y := \overline{N}(F, M)$ .  $\square$

Note that we can apply the same idea to closed non-orientable surfaces. Recall from [Rub78] that, for a closed connected 3-manifold  $M$ , a pair  $(M, F)$  is said to be a *one-sided Heegaard splitting* of  $M$  if  $F$  is a closed non-orientable surface such that  $M - F$  is an open handlebody. The cross-cap genus of this one-sided splitting is defined to that of  $F$ . The statement analogous to Corollary 4 in this context is the following.

**Corollary 5.** *A closed connected 3-manifold  $M$  admits a cross-cap genus  $g$  Heegaard splitting if and only if there exists a closed non-orientable surface  $F$  in  $M$  with cross-cap genus  $g$  such that every knot in  $M$  can be isotoped to lie within a regular neighborhood of  $F$ .*

*Proof.* Again, the statement follows immediately if we apply Theorem 18 to the neighborhood  $Y := \overline{N}(F, M)$ .  $\square$

With the affirmative resolution of Geometrization Conjecture, and in particular the Poincaré Conjecture, it is now known that a closed 3-manifold  $M$  is homeomorphic to the 3-sphere if and only if every knot in  $M$  can be *homotoped* to lie within a 3-ball contained in  $M$ . In other words, Bing's theorem (quoted as Theorem 2 in the introduction) stays valid even if we replace isotopy in the statement with homotopy. Indeed, since the affirmative answer to the Poincaré Conjecture implies

that there is no fake 3-ball, one can deduce the following: if  $M$  is a closed connected 3-manifold and  $Y$  is a connected 3-submanifold of  $M$  such that  $\partial Y$  consists only of  $S^2$ -components and that every knot in  $M$  can be homotoped to lie within  $Y$ , then  $\overline{M - Y}$  is a union of 3-balls.

It is natural to ask if the statement of our main theorem also stays valid with such modification as well, possibly with a proof that utilizes the solution to the Poincaré Conjecture. However, as it turns out, that is not the case.

**Proposition 19.** *There exists a closed connected 3-manifold  $M$  and a connected submanifold  $Y$  such that*

- *Every knot in  $M$  can be homotoped to lie within  $Y$*
- *$Y$  cannot be re-embedded into  $M$  as a Fox submanifold of  $M$ .*

*Moreover, there are an infinite number of such pair  $(M, Y)$ .*

Let us describe one source of such a manifold and its submanifold in Proposition 19. For a 3-manifold  $M$ , the rank  $r(M)$  is defined to be the smallest cardinality of generating set for  $\pi_1(M)$ . If  $M$  is a closed 3-manifold that admits a genus  $g$  Heegaard splitting, then the handlebody in the splitting gives rise to a generating set for  $\pi_1(M)$  with  $g$  elements; hence, we always have  $r(M) \leq g(M)$ . Observing  $r(M) = g(M)$  for many examples, Waldhausen [Wal78] asked if this equality holds in general.

This question was answered negatively by Boileau and Zieschang [BZ84], who found Seifert fibered manifolds  $M$  with  $r(M) = 2$  and  $g(M) = 3$ . The work of Weidmann [Wei03] provides further examples of closed manifolds  $M$  with  $r(M) < g(M)$ . Moreover, Schultens and Weidmann [SW06] recently showed that, for any non-negative integer  $n$ , there exists a closed connected 3-manifold  $M$  with  $g(M) - r(M) = n$ . It is not known whether or not there exists a hyperbolic 3-manifold with  $r(M) < g(M)$ .

*Proof of Proposition 19.* Let  $M$  be a closed 3-manifold with  $r(M) < g(M)$ . Choose a base point  $p \in M$ , and let  $\Gamma$  be the union of simple closed curves with the base point  $p$ , representing the minimal generating set of  $\pi_1(M, p)$ . Homotoping the curves if necessary, we can choose  $\Gamma$  so that  $Y := \overline{N}(\Gamma, M)$  is a handlebody of genus  $r(M)$ . If  $Y$  can be re-embedded into  $M$  as a Fox submanifold of  $M$ , we obtain a Heegaard splitting of genus  $r(M)$ . Then, we would have  $g(M) \leq r(M)$ , which clearly contradicts our assumption. The last assertion follows from [SW06].  $\square$

*Remark.* One may ask if we can characterize a closed connected 3-manifold  $M$  and a connected submanifold  $Y$  that satisfy Proposition 19.

A characterization of a closed connected 3-manifold  $M$  with  $r(M) < g(M)$  would be an important special case.

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