$$
\begin{aligned}
\int_{-\infty}^{\infty} e^{-x^2}\,dx = 2\int_{0}^{\infty} e^{-x^2}\,dx &= 2\left(\int_0^1 e^{-x^2}\,dx + \int_1^{\infty} e^{-x^2}\,dx\right) \\
&\leq 2\left(1 + \int_1^{\infty} e^{-x}\,dx\right) = 2\,\left(1 + \left[-e^{-x}\right]\big|_1^{\infty}\right) = 2\,\left(1 + e^{-1}\right).
\end{aligned}
$$

Hence $\int_{-\infty}^{\infty} e^{-x^2}\,dx$ is convergent.

We have:

$$
\begin{aligned}
\left(\int_{-\infty}^{\infty} e^{-x^2}\,dx\right)^2 &= \left(\int_{-\infty}^{\infty} e^{-x^2}\,dx\right)\left(\int_{-\infty}^{\infty} e^{-y^2}\,dy\right) \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2}e^{-y^2}\,dx\,dy \qquad \text{by Fubini} \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\,dx\,dy \\
&= \int_0^{2\pi}\int_0^{\infty} e^{-r^2}\,r\,dr\,d\theta \qquad \text{using polar coordinates} \\
&= \int_0^{2\pi}\left[\int_0^{\infty} e^{-r^2}\,r\,dr\right]d\theta \\
&= \int_0^{2\pi}\left[-\frac{e^{-r^2}}{2}\Bigg|_{r=0}^{r=\infty}\right]d\theta \\
&= \int_0^{2\pi}\left[\frac{1}{2}\right]d\theta \\
&= \pi.
\end{aligned}
$$

Refereces:

1. R. C. Buck, *Advanced Calculus*, 3rd ed., McGraw-Hill, 1978.