Research Plan

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A function $f(z)$, defined on $\mathcal{H}$ (the upper half of the complex plane), is said to be a modular form of weight k on the group Γ (of finite index in $\Gamma(1)$, the group of 2-by-2 integer matrices with determinant 1) with multiplier system v if it satisfies each of the following conditions:

1. The function must satisfy the transformation law

$$f(Mz) = v(M)(\gamma z + \delta)^k f(z), \quad \forall M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma \tag{1}$$

2. For $\Im z \geq y_0$ (for some $y_0 > 0$), f must have a Fourier expansion of the form

$$f(z) = \sum_{n=n_0}^{\infty} a_n e^{2\pi i(n+\kappa)z/\lambda}, \tag{2}$$

where $\lambda \in \mathbb{Z}^+$ is chosen such that $\begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} \in \Gamma$ and $\kappa \in [0,1)$ is chosen such that $v\begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} = e^{2\pi i \kappa}$.

3. At each rational point $q = \frac{a}{c}$, f must have a Fourier expansion of the form

$$f(z) = (z-q)^{-k} \sum_{n=n_0(q)}^{\infty} a_n(q) e^{2\pi i(n+\kappa_q)A_q^{-1}z/\lambda_q}, \tag{3}$$

where $A_q = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(1)$, $\lambda_q \in \mathbb{Z}^+$ is chosen such that $A_q \begin{pmatrix} 1 & \lambda_q \\ 0 & 1 \end{pmatrix} A_q^{-1} \in \Gamma$ and $\kappa_q \in [0,1)$ is chosen such that $v\left(A_q \begin{pmatrix} 1 & \lambda_q \\ 0 & 1 \end{pmatrix} A_q^{-1}\right) = e^{2\pi i \kappa_q}$.

Modular forms have many interesting analytic properties, and their coefficients often have equally interesting arithmetic properties. Thus, modular forms are useful analytical tools in the study of number theory.

For my dissertation, I am investigating the effects of imposing congruence restrictions on the Fourier expansion of a modular form. Given a modular form on the congruence subgroup Γ with an expansion of the form given by equation (2) (the Fourier expansion of f "at infinity"), along with integers r and t with $0 \leq r < t$, we define a new function, $f(z;r,t)$, as follows:

$$f(z;r,t) := \sum_{n \equiv r \pmod{t}}^{\infty} a_n e^{2\pi i(n+\kappa)z/\lambda}. \tag{4}$$

(When using this notation, define a_n to be zero when $n < n_0$.) This congruence-restricted Fourier expansion may be rewritten as the double-sum:

$$f(z;r,t) = \sum_{\nu \pmod{\lambda t}} e^{-2\pi i(r+\kappa)\nu/\lambda t} \sum_n e^{2\pi i(n+\kappa)(z+\frac{\nu}{t})/\lambda} \tag{5}$$

$$= \sum_{\nu \pmod{\lambda t}} e^{-2\pi i(r+\kappa)\nu/\lambda t} f(\gamma_{\nu,t} z), \tag{6}$$

where we define $\gamma_{\nu,t} := \begin{pmatrix} 1 & \frac{\nu}{t} \\ 0 & 1 \end{pmatrix}$. (That is, $\gamma_{\nu,t} z = z + \frac{\nu}{t}$.)

Several mathematicians, including Dr. Marvin Knopp (my thesis advisor), have made the observation that $f(z;r,t)$ is often a modular form on a smaller congruence subgroup $\Gamma' \subset \Gamma$. My research is motivated by this observation. For my dissertation, I am formalizing this general rule and applying the results to the study of known modular forms (and related arithmetic functions) as well as the discovery of new modular forms.

One of my primary, most frequently used results is the following lemma:

Lemma 1. Let f be a modular form as described above, with multiplier system v which satisfies the condition $v(\gamma_{a^2\nu,t} A \gamma_{-\nu,t}) = v(A)\ \forall\ \nu \in \mathbb{Z}, 0 \le \nu \le t-1$ and $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N) \cap \Gamma_{0,Mt}(t^2)$, where M denotes the minimal positive integer such that $M\kappa \in \mathbb{Z}$ (we are thus assuming here that κ is rational) and $\Gamma_{0,Mt}(t^2)$ is defined as the group of matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(t^2)$ such that $a^2 \equiv 1 \pmod{Mt}$. Then, $f(z;r,t)$ satisfies the transformation law

$$f(Az;r,t) = v(A)(cz+d)^k f(z;r,t), \quad \forall A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N) \cap \Gamma_{0,Mt}(t^2).$$

In the special case $\kappa = 0$, we may use the more general lemma:

Lemma 2. If f is a modular form (as described in the previous lemma) such that $\kappa = 0$, then

$$f(Az;r,t) = v(A)(cz+d)^k f(z;a^2 r,t), \quad \forall A \in \Gamma_0(N) \cap \Gamma_0(t^2).$$

The theta function.

We define

$$\theta(z) := \sum_{n=-\infty}^{\infty} e^{\pi i n^2 z}.$$

It is known that $\theta(2z)$ is a modular form of weight 1/2 on the group $\Gamma_0(4)$. Further, $\theta(2z)$ has the Fourier expansion,

$$\theta(2z) = \sum_{n=0}^{\infty} a(n) e^{2\pi i n z}, \text{ with } a(n) = \begin{cases} 1, & \text{if } n = 0 \\ 2, & \text{if } n \text{ is a positive integer square} \\ 0, & \text{otherwise,} \end{cases}$$

which allows us to consider the congruence restricted theta functions $\theta(2z;r,t)$. I have obtained the following useful results regarding congruence restricted theta functions:

1. $\theta(2z;r,8t)$ is a modular form of weight 1/2 on the congruence subgroup $\Gamma_1(64t^2)$.
2. Define: $f_s(z) = (\theta(2z))^s$. I have shown that $f_s(z;r,8t)$ is a modular form of weight $s/2$ on $\Gamma_1(64t^2)$, for $s \in \mathbb{Z}^+$.
3. Define: $g_s(z) = (\theta^*(2z))^s$, where

$$\theta^*(2z) = \sum_{n=0}^{\infty} b(n) e^{2\pi i n z}, \text{ with } b(n) = \begin{cases} 2, & \text{if } n \text{ is an } \textit{odd} \text{ positive integer square} \\ 0, & \text{otherwise,} \end{cases}$$

(Note that $\theta^*(2z)$ is simply $\theta(2z)$ with the sum restricted to *odd* values of n.) I have shown that $g_s(z;r,8t)$ is also a modular form of weight $s/2$ on the group $\Gamma_1(64t^2)$, for $s \in \mathbb{Z}^+$.

Regarding these results, it is important to make the following observations:

(a) For *any* value of r, $f_s(z;r,t)$ and $g_s(z;r,t)$ are both modular forms on $\Gamma_1(64t^2)$. Specifically, note that this congruence subgroup is independent of our choice of r.

(b) $f_s(z;r,t)$ is *not* the same as $(\theta(2z;r,t))^s$ (although they appear the same at first glance). The former is obtained by first taking the s^{th} power of $\theta(2z)$ and then applying a congruence restriction; the latter is found by first applying a congruence restriction and then taking the s^{th} power of the result. Thus, result #2 above is not a trivial extension of result #1.

Sums of squares.

My results with the theta function allow me to study the number theoretic functions $r_s(n)$ and $r_s^*(n)$. These functions on $\mathbb{Z}$ are defined as follows:

$$\begin{aligned} r_s(n) &= \#\{(x_1,x_2,...,x_s)\,|\,x_1^2+x_2^2+...+x_s^2=n, x_i\in\mathbb{Z}\,\forall\,1\le i\le s\} \\ r_s^*(n) &= \#\{(x_1,x_2,...,x_s)\,|\,x_1^2+x_2^2+...+x_s^2=n, x_i \text{ an } \textit{odd} \text{ integer}\,\forall\,1\le i\le s\} \end{aligned}$$

(Note that $f_s(z)$ is the generating function for $r_s(n)$, and $g_s(z)$ is the generating function for $r_s^*(n)$.)

It is shown in [1] that the ratio $\frac{r_s(8n+s)}{r_s^*(8n+s)}$ is not constant in n when $s \ge 8$. That is, when $s \ge 8$, there exists *no* constant C_s such that, for some $n_0 \in \mathbb{Z}^+$, $\frac{r_s(8n+s)}{r_s^*(8n+s)} = C_s$, $\forall n \ge n_0$. (On the other hand, in [2] it is shown that such a C_s *does* exist for $1 \le s \le 7$.) My goal is to prove a similar result for arithmetic progressions in n; that is, given certain conditions on r and t, $\frac{r_s(8n+s)}{r_s^*(8n+s)}$ is not constant on $n \equiv r \pmod t$ when $s \ge 8$. I will also investigate exceptions to this general rule, to determine conditions on r, s, and t for which there *does* exist $n_0 \in \mathbb{Z}^+$ such that $\frac{r_s(8n+s)}{r_s^*(8n+s)}$ *is* constant on $n \equiv r \pmod t$, $n \ge n_0$, when $s \ge 8$. (Experimental evidence indicates that there *are* exceptions to the general rule.)

The Dedekind eta function and the partition function.

Another research interest of mine is the partition function,

$$p(n) = \#\left\{\{x_1,x_2,...,x_k\}\,|\,x_1+x_2++x_k=n, x_i\in\mathbb{Z}^+\right\}.$$

I will investigate $p(n)$ by using congruence restrictions to study the Dedekind eta function, $\eta(z)$, which is defined:

$$\eta(z) := e^{\pi i z/12}\prod_{m=1}^{\infty}(1-e^{2\pi i m z}).$$

This function is a modular form of weight $1/2$ on $\Gamma(1)$, and it is closely related to the generating function for $p(n)$. In particular, the generating function for $p(n)$ is $e^{\pi i z/12}\frac{1}{\eta(z)}$.

The Dedekind eta function has the Fourier expansion

$$\eta(z) = \sum_{n=0}^{\infty} a_n e^{2\pi i(n+\frac{1}{24})z},$$

$$\text{where } a_n = \begin{cases} 1, & \text{if } n = m(6m\pm1) \text{ for some } m\in\mathbb{Z} \\ -1, & \text{if } n=(2m+1)(3m+1) \text{ or } n=(2m-1)(3m-1) \text{ for some } m\in\mathbb{Z} \\ 0, & \text{otherwise.} \end{cases}$$

On the other hand, $\frac{1}{\eta(z)}$ has the expansion

$$\frac{1}{\eta(z)} = \sum_{n=-1}^{\infty} p(n+1) e^{2\pi i (n+\frac{23}{24})z}.$$

I will explore the effects of imposing congruence restrictions on these expansion, in order to study $\eta(z)$, $\frac{1}{\eta(z)}$ and $p(n)$.

Expansions at rational points.

In addition to simply studying the properties of congruence-restricted modular forms "at infinity," I am also looking closely at their expansions at rational points. It is known that every modular form (as described at the beginning of this paper) has, at each point $q = \frac{a}{c} \in \mathbb{Q}$, a Fourier expansion of the form given by equation (3). It follows, then, that the same is true for congruence-restricted modular forms $f(z; r, t)$.

If $f(z)$ is a modular form on $\Gamma(1)$, we may use equation (5) to determine the expansion of $f(z; r, t)$ at $q = \frac{a}{c}$. However, the expansions derived in this manner are not immediately of the same form as is required by (3). What is interesting, then, is the observation that wheh $f(z; r, t)$ is a modular form, the two expansions *must* be the same. This fact may be used to determine the coefficients of the expansion of f at q (in terms of the coefficients of the expansion of f at ∞).

More specifically: given $q = \frac{a}{c} \in \mathbb{Q}$ (with the assumption $c > 0$), for $\nu = 0, 1, \ldots, t-1$ we choose $A_\nu = \begin{pmatrix} a_\nu & b_\nu \\ c_\nu & d_\nu \end{pmatrix} \in \Gamma(1)$ such that $\frac{a_\nu}{c_\nu} = \frac{a}{c} + \frac{\nu}{t}$. Using these matrices, we may determine the expansion of f at $\frac{a}{c} + \frac{\nu}{t}$ for each ν. Applying these expansions to equation (5), and by rewriting the exponential terms so that they will "fit" the form required by (3), we obtain:

$$(7) \quad f(z; r, t) = (z - \frac{a}{c})^{-k} \sum_{\nu \pmod t} e^{-2\pi i (r+\kappa)\nu/t} v(A_\nu) (\frac{-1}{c_\nu})^k \sum_n a_n e^{2\pi i (n+\kappa)(\frac{d\delta_\nu - t d_\nu}{ct})} e^{2\pi i (n+\kappa)(\frac{\delta_\nu}{t})^2 A^{-1} z}.$$

In order to determine the coefficients of this expansion, it remains to rewrite this sum as a sum first on n, and then on ν. This is complicated by the presence of δ_ν, which depends on a, c, ν and t.

I have made progress on solving for the coefficients of the expansion when t is prime. In this case, we have:

Case 1: $t|c$. Let μ stand for the unique integer such that $0 \le \mu \le t-1$ and $a \equiv -\frac{\mu c}{t} \pmod t$, if such an integer μ exists. Then,

$$\begin{aligned} f(z; r, t) &= \frac{1}{t}\left(z - \frac{a}{c}\right)^{-k} \times \\ &\left\{ \sum_{\substack{\nu=0 \\ \nu \ne \mu}}^{t-1} e^{-2\pi i (r+\kappa)\nu/t} (\frac{-1}{c})^k v(A_\nu) \sum_n a_n e^{2\pi i (n+\kappa)\frac{h_\nu}{t}} e^{2\pi i (n+\kappa) A^{-1} z} \right. \\ &\left. + e^{-2\pi i (r+\kappa)\mu/t} (\frac{-t}{c})^k v(A_\mu) \sum_n a_n e^{2\pi i (n+\kappa) t^2 A^{-1} z} \right\} \end{aligned}$$

Note: if $t^2|c$, then $\frac{\nu c}{t} \equiv 0 \pmod t$, $\forall 0 \le \nu \le t-1$. Therefore, the above sum on ν will run over all integers $0 \le \nu \le t-1$, and the final exponential term (involving μ) will not appear (since $a \not\equiv 0 \pmod t$.)

Case 2: $t \not| c$.

$$f(z;r,t) = \frac{1}{t}\left(z - \frac{a}{c}\right)^{-k} \sum_{\nu=0}^{t-1} e^{-2\pi i(r+\kappa)\nu/t} (\frac{-1}{ct})^k v(A_\nu) \sum_n a_n e^{2\pi i(n+\kappa)\frac{h_\nu}{t^2}} e^{2\pi i(n+\kappa)A^{-1}z/t^2}$$

(For each of the above expansions, h_ν is a constant which depends on a, c, t and ν.)

There is still much to investigate about the expansions of $f(z;r,t)$ at rational points. First of all, the above expansions must be further refined in order to deduce useful representations of their coefficients. Second, I will determine similar (though certainly more complicated) expansions in the case t is not prime. Third, I will investigate the expansions in the case that $f(z)$ is a modular form on a congruence subgroup $\Gamma_0(N)$, rather than the full modular group.

Hecke Operators.

Hecke operators are used to derive new modular forms from old ones, and they are also useful for determining properties of the coefficients of modular forms. As stated earlier, these are also the primary objectives of my study of congruence restrictions. Therefore, it makes sense to compare and contrast Hecke operators and congruence restrictions, and to investigate the effect of applying *both* operators (in either order) to a modular form.

Modular integrals.

Another research objective of mine is congruence restricted modular integrals. Modular integrals are a generalization of modular forms; in particular, if $f(z)$ is a modular integral of weight k with multiplier system v on Γ, it satisfies the transformation law

$$(cz+d)^{-k}\overline{v(M)}f(z) - f(z) = p_M(z), \forall M \in \Gamma,$$

where (c,d) is the lower row of M and $p_M(z)$ is a function defined on $\mathcal{H}$. Here, the functions $p_M(z)$ must satisfy some meaningful restriction; for example, we may require $p_M(z)$ to be a log-polynomial function, or a rational function, or (in the most general case) an element of $\mathcal{P}$, where we define

$$\mathcal{P} := \left\{\phi(z) \text{ holomorphic in } \mathcal{H}\text{: } |\phi(z)| \le K(|z|^\alpha + y^{-\beta})\right\}$$

for some constants K, α and β. The functions $p_M(z)$ are called the *period functions* of $f(z)$. (Note that, if $f(z)$ is a modular form, then $p_M(z) = 0 \ \forall M \in \Gamma$.) If $p_S(z) \equiv 0$ (where $S = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$), then $f(z)$ is periodic and, therefore, has a Fourier expansion similar to that of a modular form. Thus, we may study the effect of congruence restrictions on the Fourier expansions of modular integrals. My objective will be to use congruence restrictions to derive new modular integrals from old ones and to study the coefficients of known modular integrals.

References

[1] P.T. Bateman, B. Datskovsky and M.I. Knopp, "Sums of Squares and Preservation of Modularity Under Congruence Restrictions," to appear.

[2] P.T. Bateman and M.I. Knopp, "Some New Old-Fashioned Modular Identities," *The Ramanujan Journal* **2** (1998), 247-269.