Group actions and rational ideals

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Overview

- Rational ideals: definition and connection with irreducible representations

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- Actions of algebraic groups: brief reminder of some basics

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- The main result on rational ideals

- Some history . . . if time

References

- *“Group actions and rational ideals”*,
  Algebra and Number Theory **2** (2008), 467-499

- *“Algebraic group actions on noncommutative spectra”*,
  posted at http://arXiv.org/abs/0809.5205

Both articles & the **pdf file of this talk** available on my web page:

http://math.temple.edu/~lorenz/

Part I: Rational Ideals

Definition of rational ideals

“Dixmier-Mœglin equivalence”

Under favorable circumstances, rational ideals are the same as primitive ideals, that is, kernels of irreducible representations.

In detail . . .

Definition of rational ideals

Notation:	$\mathbb{k}$	some algebraically closed base field
(entire talk)	R	an associative $\mathbb{k}$-algebra (with 1)

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"Herz"
"core"
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Definition:

- The extended centroid of R is defined by

$$\mathcal{C}(R) = \mathcal{Z}\,\mathrm{Q_r}(R)$$

Here $\mathrm{Q_r}$ is the right Amitsur-Martindale quotient ring and $\mathcal{Z}$ denotes the center.

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  Here $\mathrm{Q_r}$ is the right Amitsur-Martindale quotient ring and $\mathcal{Z}$ denotes the center.

- If $P \in \operatorname{Spec} R$ then $\mathcal{C}(R/P)$ is a $\Bbbk$-field. We call P rational if $\mathcal{C}(R/P) = \Bbbk$.

Origins of the Amitsur-Martindale quotient ring

I will not discuss the definition of $\mathrm{Q_r}(R)$. Here are the original sources:

for prime rings R:

W. S. Martindale, III, *Prime rings satisfying a generalized polynomial identity*, J. Algebra **12** (1969), 576–584.

for general R:

S. A. Amitsur, *On rings of quotients*, Symposia Math., Vol. VIII, Academic Press, London, 1972, pp. 149–164.

Examples

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- For R **semiprime right Goldie**,

$$\mathrm{Q_r}(R) = \{q \in \mathrm{Q_{cl}}(R) \mid qI \subseteq R \text{ for some } I \lhd R \text{ with } \mathrm{ann}_R\, I = 0\}$$

In particular,

$$\boxed{\mathcal{C}(R) = \mathcal{Z}\mathrm{Q_{cl}}(R)}$$

classical quotient ring of R

Examples

- $R = U(\mathfrak{g})/I$ a semiprime image of the enveloping algebra of a finite-dimensional Lie algebra $\mathfrak{g}$. Then

$$\mathrm{Q_r}(R) = \{ \text{ ad } \mathfrak{g}\text{-finite elements of } \mathrm{Q_{cl}}(R) \}$$

Connection with irreducible representations

Given an irreducible representation $\rho\colon R \to \operatorname{End}_{\Bbbk}(V)$, let $P = \operatorname{Ker}\rho$ be the corresponding primitive ideal of R.

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- There **always** is an embedding of $\Bbbk$-fields

$$\mathcal{C}(R/P) \hookrightarrow \mathcal{Z}\left(\operatorname{End}_R(V)\right)$$

- **Typically**, $\operatorname{End}_R(V) = \Bbbk$ (“weak Nullstellensatz”); in this case

$$\text{primitive} \quad \Rightarrow \quad \text{rational}$$

Examples

The weak Nullstellensatz holds for

- R any affine $\Bbbk$-algebra, $\Bbbk$ uncountable — Amitsur
- R an affine PI-algebra — Kaplansky
- $R = U(\mathfrak{g})$ — "Quillen's Lemma"
- $R = \Bbbk\Gamma$ with Γ polycyclic-by-finite — Hall, L.
- many quantum groups: $\mathcal{O}_q(\Bbbk^n)$, $\mathcal{O}_q(M_n(\Bbbk))$, $\mathcal{O}_q(G)$, $\ldots$

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In fact, in all these examples except the first, it has been shown that the **Dixmier-Mœglin equivalence** holds (under mild restrictions on $\Bbbk$ or q):

primitive $\Leftrightarrow$ rational

Part II: Algebraic Groups

Definition

There an anti-equivalence of categories

$$\left\{\begin{array}{l}\textbf{affine algebraic}\\ \textbf{groups }/\mathbb{k}\end{array}\right\} \simeq \left\{\begin{array}{l}\textbf{comm. affine reduced}\\ \textbf{Hopf }\mathbb{k}\textbf{-algebras}\end{array}\right\}$$

$$G = \mathrm{Hom}_{\mathbb{k}\text{-alg}}(\mathbb{k}[G], \mathbb{k}) \quad \longleftrightarrow \quad \mathbb{k}[G]$$

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$$G = \mathrm{Hom}_{\Bbbk\text{-alg}}(\Bbbk[G], \Bbbk) \longleftrightarrow \Bbbk[G]$$

Equivalently, affine algebraic groups are precisely the closed subgroups of $\mathbf{GL}_n$ for some n:

$\mathbf{GL}_n, \mathbf{SL}_n, \mathbf{T}_n, \mathbf{U}_n, \mathbf{D}_n, \mathbf{O}_n, \dots$
all finite groups

Rational representations

For any affine algebraic $\mathbb{k}$-group G,

$$G\text{-modules} \quad \equiv \quad \mathbb{k}[G]\text{-comodules}$$

Comodule structure map

$$\begin{aligned} \Delta_M \colon M &\longrightarrow M \otimes \mathbb{k}[G] \\ m &\longmapsto \sum m_0 \otimes m_1 \end{aligned}$$

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We obtain a linear representation $G \to \mathbf{GL}(M)$ by

$$g.m = \sum m_0 m_1(g)$$

Representations arising in the way are called rational.

Rational representations

- Rational rep^s are **locally finite**. In fact, they can also be characterized as the linear rep^s $G \to \mathbf{GL}(M)$ that are

  (a) locally finite and

  (b) for each finite-dimensional G-stable subspace $V \subseteq M$, the resulting map $G \to \mathbf{GL}(V)$ is a morphism of alg. groups.

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- Tensor products of rational rep$^{\text{s}}$ of G are again rational. Similarly for sums, subrep$^{\text{s}}$ and homomorphic images.

Part III: Main Result on Rational Ideals

G-prime and G-rational ideals

Let G be an arbitrary group acting on R by $\Bbbk$-algebra auto$^{\text{s}}$.

G-prime and G-rational ideals

Let G be an arbitrary group acting on R by $\Bbbk$-algebra auto^s.

Have induced G-actions on

- $\{$ideals of $R\}$
- $\operatorname{Spec} R = \{$prime ideals of $R\}$
- $\operatorname{Rat} R = \{$rational ideals of $R\}$
- $\operatorname{Prim} R = \{$primitive ideals of $R\}$
- $\mathrm{Q_r}(R)$ and $\mathcal{C}(R) = \mathcal{Z}\,\mathrm{Q_r}(R)$

Notation: $G\backslash \operatorname{Spec} R$ will denote the set of G-orbits in $\operatorname{Spec} R$, and similarly for other G-sets.

G-prime and G-rational ideals

Definition: A proper G-stable ideal $I \lhd R$ is called

- G-prime if $AB \subseteq I$ for G-stable ideals $A, B \lhd R$ implies that $A \subseteq I$ or $B \subseteq I$.
- G-rational if I is G-prime and $\mathcal{C}(R/I)^G = \Bbbk$.

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- G-rational if I is G-prime and $\mathcal{C}(R/I)^G = \Bbbk$.

Put $G\text{-}\operatorname{Spec} R = \{G\text{-prime ideals of } R\}$ and similarly for $G\text{-}\operatorname{Rat} R$.

Prop[n]: *The assignment* $P \mapsto \bigcap_{g \in G} g.P$ *yields maps* $G\backslash \operatorname{Spec} R \to G\text{-}\operatorname{Spec} R$ *and* $G\backslash \operatorname{Rat} R \to G\text{-}\operatorname{Rat} R$.

Rational actions of algebraic groups

Now assume that G is an affine alg. $\Bbbk$-group acting rationally on the $\Bbbk$-algebra R; so we have a rational rep$^{\text{n}}$

$$G \longrightarrow \mathrm{Aut}_{\Bbbk\text{-alg}}(R) \subseteq \mathbf{GL}(R)$$

Equivalently, R is a $\Bbbk[G]$-comodule algebra.

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Equivalently, R is a $\Bbbk[G]$-comodule algebra.

Theorem:

$$\begin{array}{ccc} G\backslash \operatorname{Rat} R & \xrightarrow{\text{bij.}} & G\text{-}\operatorname{Rat} R \\ \Psi & & \Psi \\ G.P & \longmapsto & \bigcap_{g \in G} g.P \end{array}$$

Some comments

- $G \backslash \operatorname{Spec} R \to G\text{-}\operatorname{Spec} R$ is surjective (easy, even under more general circumstances) but it is rarely injective (exactly if all primes of R are stable under the connected component of G).

- The Th[m] easily reduces to the case where G is connected. In this case, there is the following result on algebraic groups, due to **Vonessen** (1998) and Abe & Kanno (1959).

Prop[n]: *Let G act on $\Bbbk(G)$ via ρ_r and let F be a G-stable $\Bbbk$-subfield of $\Bbbk(G)$. Let $\operatorname{Hom}_G(F, \Bbbk(G))$ denote the collection of all G-equivariant $\Bbbk$-algebra homomorphisms $\phi\colon F \to \Bbbk(G)$. Then the G-action on $\operatorname{Hom}_G(F, \Bbbk(G))$ that is given by $g.\phi = \rho_\ell(g) \circ \phi$ is transitive.*

Here, ρ_r and ρ_ℓ denote the right and left regular actions of G on its function field $\Bbbk(G)$.

- The **fibre** over any $I \in G\text{-}\operatorname{Rat} R$ is in G-equivariant bijection with $\operatorname{Hom}_G(\mathcal{C}(R/I), \Bbbk(G))$, and this set is nonempty. The Th[m] follows.

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Part IV: Some History

Jacques Dixmier (* 1924)

from P. Halmos, “I Have A Photographic Memory”

- former member of Bourbaki
- Ph.D. advisor of A. Connes
- author of several highly influential monographs:

Les algèbres d’opérateurs dans l’espace hilbertien: algèbres de von Neumann, Gauthier-Villars, 1957

Les C^*-algèbres et leurs représentations, Gauthier-Villars, 1969

Algèbres enveloppantes, Gauthier-Villars, 1974

Some milestones

Dixmier's Problem # 11 (from: *Algèbres enveloppantes*, 1974)

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10. On suppose que $\mathrm{tr}\,\mathrm{ad}\,x = 0$ pour tout $x \in \mathfrak{g}$. Est ce que $Z(\mathfrak{g}) \neq k$?

11. Soient $\mathfrak{k}$ un idéal de $\mathfrak{g}$, I un idéal primitif de $U(\mathfrak{g})$. Les propriétés suivantes sont-elles vraies : (*a*) il existe un idéal primitif de $U(\mathfrak{k})$ générique pour $U(\mathfrak{k}) \cap I$; (*b*) deux tels idéaux sont conjugués par le groupe adjoint algébrique de $\mathfrak{g}$; (*c*) soit L un tel idéal; il existe une représentation simple σ de $\mathfrak{k}$ de noyau L, et une représentation simple ρ de $\mathfrak{st}(\sigma, \mathfrak{g})$, telles que $\rho \mid \mathfrak{k}$ soit un multiple de σ et que $\mathrm{ind}(\rho, \mathfrak{g})$ soit simple de noyau I. *Cf.* 4.5.9, 5.4.3, 5.4.4, 5.6.5.

Some milestones

- Problem 11 for $\mathfrak{k}$ solvable, $\operatorname{char} \Bbbk = 0$

Dixmier, *Sur les idéaux génériques dans les algèbres enveloppantes*, Bull. Sci. Math. (2) **96** (1972), 17–26. ⇝ existence

Borho, Gabriel, Rentschler, *Primideale in Einhüllenden auflösbarer Lie-Algebren*, Springer Lect. Notes in Math. 357 (1973). ⇝ uniqueness

- Theorem 1 under Goldie hypotheses, $\operatorname{char} \Bbbk = 0$: **Mœglin & Rentschler**

Orbites d'un groupe algébrique dans l'espace des idéaux rationnels d'une algèbre enveloppante, Bull. Soc. Math. France **109** (1981), 403–426.

Sur la classification des idéaux primitifs des algèbres enveloppantes, Bull. Soc. Math. France **112** (1984), 3–40.

Sous-corps commutatifs ad-stables des anneaux de fractions des quotients des algèbres enveloppantes; espaces homogènes et induction de Mackey, J. Funct. Anal. **69** (1986), 307–396.

Idéaux G-rationnels, rang de Goldie, preprint, 1986.

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Some milestones

- Theorem 1 under Goldie hypotheses, $\mathrm{char}\,\Bbbk$ arbitrary:
  N. Vonessen

Actions of algebraic groups on the spectrum of rational ideals,
J. Algebra **182** (1996), 383–400.

Actions of algebraic groups on the spectrum of rational ideals. II,
J. Algebra **208** (1998), 216–261.

Rudolf Rentschler (PhD 1967 Munich)