Multiplicative Invariant Theory

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Overview

- **Introduction** to multiplicative invariants: definitions, examples, . . .

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- **Regularity**: reflection groups and semigroup algebras

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- **Introduction** to multiplicative invariants: definitions, examples, . . .
- **Regularity**: reflection groups and semigroup algebras
- The **Cohen-Macaulay property**: reminders on CM rings and some results on multiplicative invariants

Part I: Introduction

Multiplicative Invariants

- Given: a group G and a G-lattice $L \cong \mathbb{Z}^n$; so

$$G \longrightarrow \mathrm{GL}(L) \cong \mathrm{GL}_n(\mathbb{Z})$$

an integral representation of G

Multiplicative Invariants

- Given: a group G and a G-lattice $L \cong \mathbb{Z}^n$; so

$$G \longrightarrow \mathrm{GL}(L) \cong \mathrm{GL}_n(\mathbb{Z})$$

- Choose a base ring $\Bbbk$ and form the group algebra

$$\Bbbk[L] = \bigoplus_{m \in L} \Bbbk \mathbf{x}^m \cong \Bbbk[x_1^{\pm 1}, \dots, x_n^{\pm 1}] \,, \qquad \mathbf{x}^m \mathbf{x}^{m'} = \mathbf{x}^{m+m'}$$

  The G-action on L extends uniquely to a "multiplicative" action by $\Bbbk$-algebra automorphisms on $\Bbbk[L]$:

$$g(\mathbf{x}^m) = \mathbf{x}^{g(m)} \qquad (g \in G, m \in L)$$

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  The G-action on L extends uniquely to a "multiplicative" action by $\Bbbk$-algebra automorphisms on $\Bbbk[L]$.

- The multiplicative invariant algebra is

$$\Bbbk[L]^G = \{f \in \Bbbk[L] \mid g(f) = f \ \forall g \in G\}$$

Example #1

Multiplicative inversion in rank 2**:**
($\mathbb{k} = \mathbb{Z}$)

$G = \langle g \mid g^2 = 1 \rangle$
$L = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2$
action: $g(e_i) = -e_i$

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Putting $x_i = \mathbf{x}^{e_i}$ we have:

$$\mathbb{Z}[L] = \mathbb{Z}[x_1^{\pm 1}, x_2^{\pm 1}] \quad \text{with } g(x_i) = x_i^{-1}$$

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Straightforward calculation gives

$$\mathbb{Z}[L]^G = \mathbb{Z}[\xi_1, \xi_2] \oplus \eta\mathbb{Z}[\xi_1, \xi_2]$$

with $\xi_i = x_i + x_i^{-1}$ and $\eta = x_1 x_2 + x_1^{-1} x_2^{-1}$; they satisfy

$$\eta\xi_1\xi_2 = \eta^2 + \xi_1^2 + \xi_2^2 - 4$$

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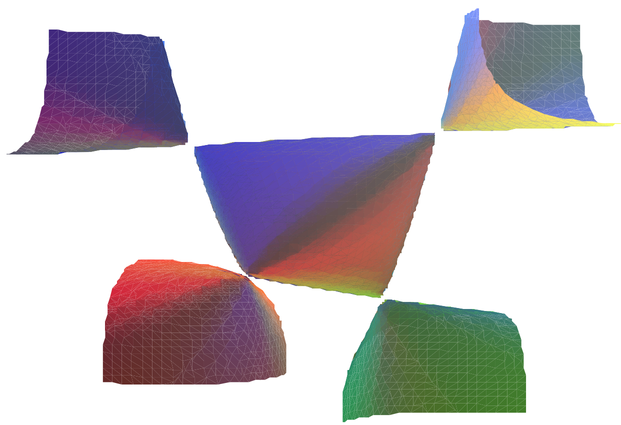
Hence: $\mathbb{Z}[L]^G \cong \mathbb{Z}[x, y, z]/(x^2 + y^2 + z^2 - xyz - 4)$



Example #1$'$: linear version

Linear inversion in rank 2:

$G = \langle g \mid g^2 = 1 \rangle$
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action: $g(e_i) = -e_i$

Example #1′: linear version

Linear inversion in rank 2:

$G = \langle g \mid g^2 = 1 \rangle$
$L = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2$
action: $g(e_i) = -e_i$

Now: $\mathsf{S}(L) = \mathbb{Z}[x_1, x_2]$ with $g(x_i) = -x_i$

Example #1$'$: linear version

Linear inversion in rank 2:

$G = \langle g \mid g^2 = 1 \rangle$
$L = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2$
action: $g(e_i) = -e_i$

One obtains:

$$\mathsf{S}(L)^G = \mathbb{Z}[\xi_1, \xi_2] \oplus \eta\mathbb{Z}[\xi_1, \xi_2] \qquad (\xi_i = x_i^2,\ \eta = x_1x_2)$$

Relation: $\eta^2 = \xi_1\xi_2$

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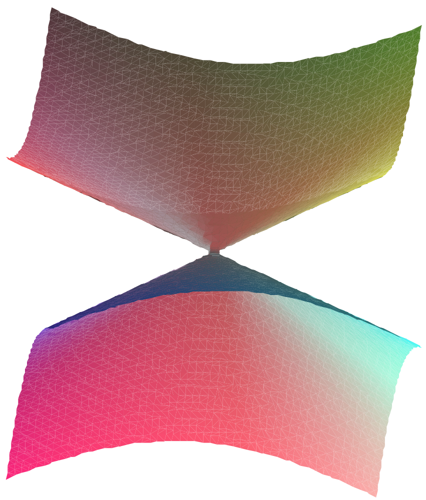

Some Special Features

Back to general multiplicative actions:

L	a G-lattice
$\Bbbk$	a commutative base ring
$\Bbbk[L]$	the group algebra

Some Special Features

Multiplicative invariants have a $\mathbb{Z}$-structure:

a $\Bbbk$-basis of $\Bbbk[L]^G$ is given by the distinct orbit sums

$$\mathrm{orb}(m) := \sum_{m' \in G(m)} \mathbf{x}^{m'} \qquad (m \in L)$$

$$\Downarrow$$

$$\Bbbk[L]^G = \Bbbk \otimes_{\mathbb{Z}} \mathbb{Z}[L]^G$$

Some Special Features

It suffices to consider finite groups:

each $\mathrm{orb}(m)$ is supported on

$$L_{\text{fin}} = \{m \in L \mid [G : G_m] < \infty\}$$

stabilizer of $m \in L$

G acts on L_{fin} through the finite quotient $\mathcal{G} = G/\operatorname{Ker}_G(L_{\text{fin}})$. Thus:

$$\Bbbk[L]^G = \Bbbk[L_{\text{fin}}]^{\mathcal{G}}$$

Some Special Features

In particular, $\mathbb{k}[L]^G$ is always affine/$\mathbb{k}$
(Hilbert # 14 ok).

On the other hand . . .

Some Special Features

In general, $\Bbbk[L]$ has no grading (connected) that is preserved by the action of G.

$\rightsquigarrow$ computational theory not yet highly developed

$\exists$ some GAP & MAGMA-programs (L., Marc Renault)

Finite Linear Groups

Jordan (1880): $\mathrm{GL}_n(\mathbb{Z})$ has only finitely many finite subgroups up to conjugacy.

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$\rightsquigarrow$ there are only finitely many multiplicative invariant algebras $\Bbbk[L]^G$ (up to $\cong$) with $\operatorname{rank} L$ bounded

Finite Linear Groups

Jordan (1880): $\mathrm{GL}_n(\mathbb{Z})$ has only finitely many finite subgroups up to conjugacy.

Minkowski (1887): The least common multiple of their orders is given by

$$M_n = \prod_p p^{\left\lfloor \frac{n}{p-1} \right\rfloor + \left\lfloor \frac{n}{p(p-1)} \right\rfloor + \left\lfloor \frac{n}{p^2(p-1)} \right\rfloor + \cdots}$$

Finite Linear Groups

n	# fin. $\mathcal{G} \leq \mathrm{GL}_n(\mathbb{Z})$ (up to conj.)	# max'l $\mathcal{G}$ (up to conj.)	M_n
1	2	1	2
2	13	2	24
3	73	4	48
4	710	9	5760
5	6079	17	11520
6	85311	39	2903040

Pioneers of MIT

- **Bourbaki**: *"Invariants exponentiels"* (Chap. VI § 3 of *Groupes et algèbres de Lie*, 1968)

$$R(\mathfrak{g}) \cong \mathbb{Z}[\Lambda]^{\mathcal{W}} \cong \mathbb{Z}[x_1, \ldots, x_{\operatorname{rank}\mathfrak{g}}]$$

where $R(\mathfrak{g})$ = representation ring of a semisimple Lie algebra $\mathfrak{g}$, Λ = weight lattice of $\mathfrak{g}$, and $\mathcal{W}$ = Weyl group.

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- **Steinberg, Richardson** (1970s)

- "Δ-methods" for group rings: **Passman**, **Zalesskiĭ**, **Roseblade**, **Dan Farkas** $\rightsquigarrow$ "multiplicative invariants" (mid 1980s)

Part II: Regularity

Regularity at 1

Notations:

$\mathcal{G}$	a finite group
$L \cong \mathbb{Z}^n$	a faithful $\mathcal{G}$-lattice
$\Bbbk = \overline{\Bbbk}$	a field with $\operatorname{char} \Bbbk \nmid \lvert\mathcal{G}\rvert$

Will explain the following result . . .

Regularity at 1

Theorem 1 *TFAE* (1) $\mathbb{k}[L]^{\mathcal{G}}$ *is regular at* $\pi(1)$
(2) $\mathcal{G}$ *acts as a reflection group on* L
(3) $\mathbb{k}[L]^{\mathcal{G}} = \mathbb{k}[M]$ *is a semigroup algebra with* $\varepsilon(M) \subseteq \mathbb{k}^*$

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Here,

$$\begin{array}{ccc} X = \operatorname{Spec} \Bbbk[L] & \xrightarrow{\pi} & X/\mathcal{G} = \operatorname{Spec} \Bbbk[L]^{\mathcal{G}} \\ \Cup & & \\ 1 = \operatorname{Ker} \varepsilon & & \end{array}$$

where $\varepsilon \colon \Bbbk[L] \longrightarrow \Bbbk$ is the counit: $\varepsilon(\mathbf{x}^m) = 1$ for all $m \in L$

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(1) $\Rightarrow$ (2) uses linearization: Put $\mathfrak{E} = \operatorname{Ker} \varepsilon$. Then

$$\begin{array}{ccc} L_{\Bbbk} = L \otimes \Bbbk & \stackrel{\sim}{\longrightarrow} & \mathfrak{E}/\mathfrak{E}^2 \\ m \otimes 1 & \mapsto & \mathbf{x}^m - 1 + \mathfrak{E}^2 \end{array}$$

leads to $\mathsf{S}\widehat{(L_{\Bbbk})}^{\mathcal{G}}_{\pi(0)} \cong \widehat{\Bbbk[L]}^{\mathcal{G}}_{\pi(1)}$. Now use the S-T-C Theorem.

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(3) $\Bbbk[L]^{\mathcal{G}} = \Bbbk[M]$ *is a semigroup algebra with* $\varepsilon(M) \subseteq \Bbbk^*$

<u>(2) $\Rightarrow$ (3)</u> uses root systems: $\exists$ root system Φ so that

$$\mathbb{Z}\Phi \subseteq L \subseteq \Lambda(\Phi) \qquad \text{with} \quad \mathcal{G} = \mathcal{W}(\Phi)$$

Use **Bourbaki's Thm:** $\mathbb{Z}[\Lambda(\Phi)]^{\mathcal{W}(\Phi)}$ is a polynomial algebra.

Regularity at 1

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<u>(3) $\Rightarrow$ (1)</u> uses torus actions:

(3) $\Leftrightarrow$ $X/\mathcal{G} = \operatorname{Spec} \Bbbk[L]^{\mathcal{G}}$ is an affine toric variety so that $\pi(1)$ belongs to the open torus orbit

This implies (1).

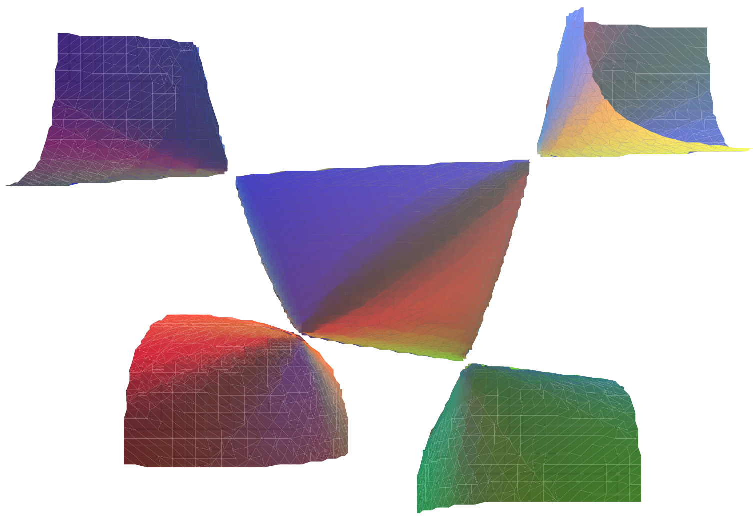
Example #1 revisited

Recall: **multiplicative inversion** (rank 2)

$\mathcal{G} = \langle g \mid g^2 = 1 \rangle$
$L = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2$
action: $g(e_i) = -e_i$

$$\rightsquigarrow \qquad \Bbbk[L]^{\mathcal{G}} \cong \Bbbk[x,y,z]/(x^2+y^2+z^2-xyz-4)$$

$\Bbbk[L]^{\mathcal{G}}$ is **not** a semi-group algebra:



Example #2: U_n and the root lattice A_{n-1}

Notation: $U_n = \bigoplus_1^n \mathbb{Z}e_i \cong \mathbb{Z}^n$

$A_{n-1} = \{\sum_i z_i e_i \in U_n \mid \sum_i z_i = 0\} \cong \mathbb{Z}^{n-1}$

$\mathcal{S}_n$-action: $\sigma(e_i) = e_{\sigma(i)} \ (\sigma \in \mathcal{S}_n)$

Note: $\mathcal{S}_n$ acts as a reflection group; transpositions are reflections

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$\mathcal{S}_n$-action: $\sigma(e_i) = e_{\sigma(i)}$ $(\sigma \in \mathcal{S}_n)$

Put $x_i = \mathbf{x}^{e_i} \in \Bbbk[U_n]$; so $\sigma(x_i) = x_{\sigma(i)}$ for $\sigma \in \mathcal{S}_n$. Then

$$\Bbbk[U_n] = \Bbbk[x_1^{\pm 1}, \ldots, x_n^{\pm 1}] = \Bbbk[x_1, \ldots, x_n][s_n^{-1}] \ ,$$

where $s_n = \prod_1^n x_i$ is the n^{th} elementary symmetric polynomial.

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$A_{n-1} = \{\sum_i z_i e_i \in U_n \mid \sum_i z_i = 0\} \cong \mathbb{Z}^{n-1}$

$\mathcal{S}_n$-action: $\sigma(e_i) = e_{\sigma(i)}$ $(\sigma \in \mathcal{S}_n)$

$$\therefore \quad \mathbb{k}[U_n]^{\mathcal{S}_n} = \mathbb{k}[x_1, \ldots, x_n][s_n^{-1}]^{\mathcal{S}_n}$$
$$= \mathbb{k}[x_1, \ldots, x_n]^{\mathcal{S}_n}[s_n^{-1}]$$
$$= \mathbb{k}[s_1, \ldots, s_{n-1}, s_n^{\pm 1}]$$
$$\cong \mathbb{k}[\mathbb{Z}_+^{n-1} \oplus \mathbb{Z}]$$

elem. symmetric poly's

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$\mathcal{S}_n$-action: $\sigma(e_i) = e_{\sigma(i)} \ (\sigma \in \mathcal{S}_n)$

Now,

$$\Bbbk[A_{n-1}] = \Bbbk[U_n]_0 \ ,$$

the degree 0-component for the ($\mathcal{S}_n$-stable) “total degree” grading of $\Bbbk[U_n] = \Bbbk[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$.

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$\mathcal{S}_n$-action: $\sigma(e_i) = e_{\sigma(i)}$ $(\sigma \in \mathcal{S}_n)$

Get $\mathbb{k}[A_{n-1}]^{\mathcal{S}_n} = \mathbb{k}[U_n]_0^{\mathcal{S}_n} = \mathbb{k}[s_1, \ldots, s_{n-1}, s_n^{\pm 1}]_0$; so

$$\mathbb{k}[A_{n-1}]^{\mathcal{S}_n} \cong \mathbb{k}[M]$$

with

$$M = \left\{ (t_1, \ldots, t_{n-1}) \in \mathbb{Z}_+^{n-1} \mid \sum i t_i \in n\mathbb{Z} \right\}$$

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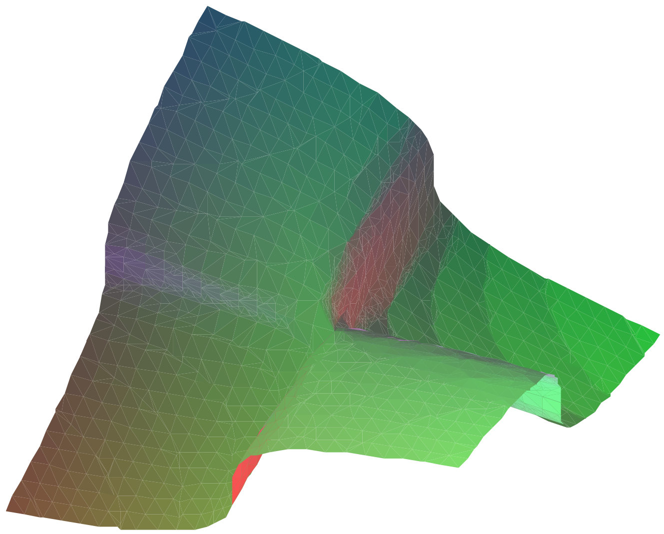
$\mathcal{S}_n$-action: $\sigma(e_i) = e_{\sigma(i)}$ $(\sigma \in \mathcal{S}_n)$

$\mathbb{C}[A_{n-1}]^{\mathcal{S}_n}$ is **not** regular:
($n > 2$; picture for $n = 3$)



Regularity

Here is the global version of Theorem 1

(same notations and hypotheses)

Regularity

Theorem 1′ *TFAE* (1) $\Bbbk[L]^{\mathcal{G}}$ *is regular*
(2) $\mathcal{G}$ *acts as a reflection group on* L *and* $H^1(\mathcal{G}/\mathcal{D}, L^{\mathcal{D}}) = 0$
(3) $\Bbbk[L]^{\mathcal{G}} \cong \Bbbk[\mathbb{Z}_+^r \oplus \mathbb{Z}^s]$
(4) $\exists$ *root system* Φ *s.t.* $L/L^{\mathcal{G}} \cong \Lambda(\Phi)$ *and* $\mathcal{G} = \mathcal{W}(\Phi)$

Here, $\mathcal{D}$ is the subgroup of $\mathcal{G}$ that is generated by the "diagonalizable" reflections, conjugate in $\mathrm{GL}(L)$ to

$$d = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}$$

Regularity

n	# finite $\mathcal{G} \leq \mathrm{GL}_n(\mathbb{Z})$ (up to conjugacy)	# reflection groups $\mathcal{G}$ (up to conjugacy)	# cases with $\Bbbk[L]^{\mathcal{G}}$ regular
2	13	9	7
3	73	29	18
4	710	102	51

Part III: The Cohen-Macaulay Property

Reminder: CM Rings

- **Hypotheses:** R a comm. noetherian ring
  $\mathfrak{a}$ an ideal of R

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- **Hypotheses:** R a comm. noetherian ring
  $\mathfrak{a}$ an ideal of R
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(Zariski) topology
dimension theory

(homological) algebra

- **Def:** R is **Cohen-Macaulay** iff equality holds for all (maximal) ideals $\mathfrak{a}$

Some Examples of CM Rings

- **Standard example:** R an affine domain/PID $\Bbbk$, finite / some polynomial subalgebra $P = \Bbbk[x_1, \ldots, x_n]$. Then:

$$R \text{ CM} \Leftrightarrow R \text{ is free over } P$$

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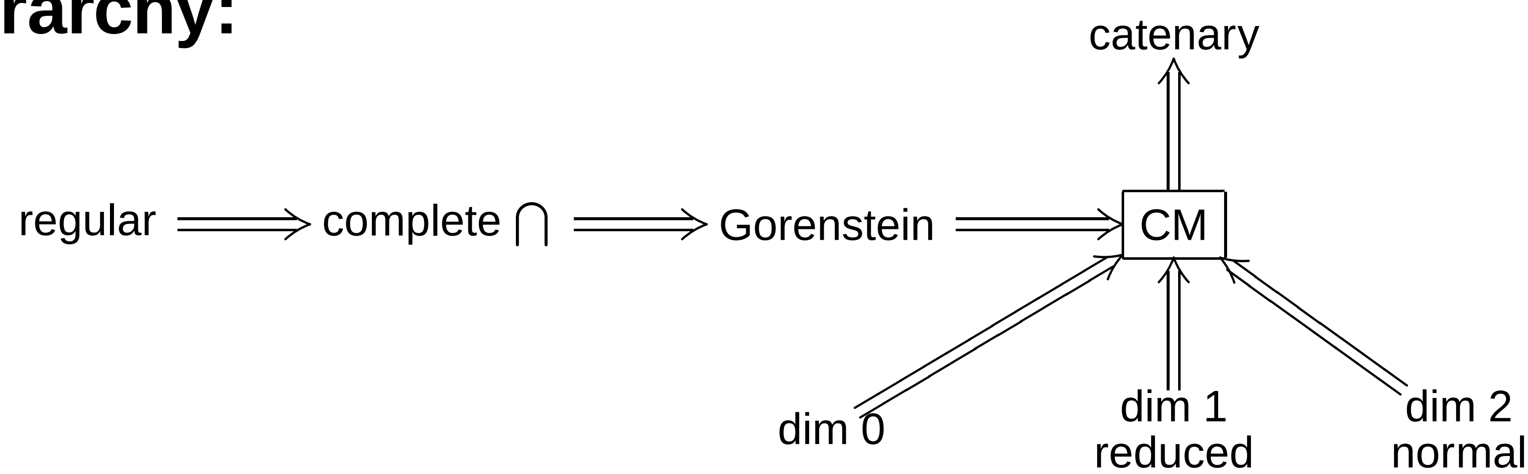
- **Hierarchy:**



Invariant Rings

Hypotheses: R a CM ring
$\mathcal{G}$ a finite group acting on R

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If the **trace map** $R \to R^{\mathcal{G}}$, $r \mapsto \sum_{\mathcal{G}} g(r)$, is epi ("non-modular case") then $R^{\mathcal{G}}$ is CM; otherwise usually not.

Invariant Rings

Hypotheses:

R	a CM ring
$\mathcal{G}$	a finite group acting on R

If the **trace map** $R \to R^{\mathcal{G}}$, $r \mapsto \sum_{\mathcal{G}} g(r)$, is epi ("non-modular case") then $R^{\mathcal{G}}$ is CM; otherwise usually not.

Here is a **necessary condition** . . .

Invariant Rings

Hypotheses:

R a CM ring

$\mathcal{G}$ a finite group acting on R

$\mathcal{R}_k = \{ k$-reflections on $R \}$

Assume R noetherian $/R^{\mathcal{G}}$

automorphisms belonging to the inertia group of some prime of height $\leq k$

Proposition
(L. - Pathak)

$R^{\mathcal{G}}$ *CM* & $H^i(\mathcal{G}, R) = 0 \ (0 < i < k)$

$$\Rightarrow \quad \mathrm{res}\colon H^k(\mathcal{G}, R) \hookrightarrow \prod_{\mathcal{H} \subseteq \mathcal{R}_{k+1}} H^k(\mathcal{H}, R)$$

Invariant Rings

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Note: The $(H^i = 0)$-cond$^{\mathrm{n}}$ is vacuous for $k = 1$ $\rightsquigarrow$ bireflections.

Multiplicative Invariants: CM-property

Notations: $\mathcal{G}$ is a finite group $\neq 1$
L a $\mathcal{G}$-lattice, WLOG faithful

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So $\mathcal{G} \hookrightarrow \mathrm{GL}(L), g \mapsto g_L$. In this setting,

$g \in \mathcal{G}$ is a k-reflection on $\Bbbk[L]$	$\iff$	$\mathrm{rank}(g_L - \mathrm{Id}_L) \leq k$

"g is a k-reflection on L" — or on $L \otimes_{\mathbb{Z}} \mathbb{Q}$

Multiplicative Invariants: CM-property

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Theorem 2 (L, TAMS '06) *If $\mathbb{Z}[L]^{\mathcal{G}}$ is CM then all $\mathcal{G}_m/\mathcal{R}^2(\mathcal{G}_m)$ for $m \in L$ are perfect groups, but not all $\mathcal{G}_m$ are.*

subgroup gen. by bireflections on L

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Corollary ("3-copies conjecture") *$\mathbb{Z}[L^{\oplus r}]^{\mathcal{G}}$ is never CM for $r \geq 3$.*

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Note that the conclusions of Theorem 2 only refer to the rational type of L. In fact . . .

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Notations: $\mathcal{G}$ is a finite group $\neq 1$
L a $\mathcal{G}$-lattice, WLOG faithful

Theorem 2 (L, TAMS '06) *If $\mathbb{Z}[L]^{\mathcal{G}}$ is CM then all $\mathcal{G}_m/\mathcal{R}^2(\mathcal{G}_m)$ for $m \in L$ are perfect groups, but not all $\mathcal{G}_m$ are.*

Proposition *If $\Bbbk[L]^{\mathcal{G}}$ is CM then so is $\Bbbk[L']^{\mathcal{G}}$ for any $\mathcal{G}$-lattice L' so that $L' \otimes \mathbb{Q} \cong L \otimes \mathbb{Q}$.*

Example: $\mathcal{S}_n$-lattices

What are the $\mathcal{S}_n$-lattices L such that $\mathbb{Z}[L]^{\mathcal{S}_n}$ is CM ?

Example: $\mathcal{S}_n$-lattices

We know:

- only the structure of $L_{\mathbb{Q}} = L \otimes_{\mathbb{Z}} \mathbb{Q}$ matters (Proposition)
- $\mathcal{S}_n$ must act as a bireflection group on L (Theorem 2), and hence on all simple constituents of $L_{\mathbb{Q}}$

Example: $\mathcal{S}_n$-lattices

Classification results of irreducible finite linear groups containing a bireflection (Huffman and Wales, 70s) imply, for $n \geq 7$:

$$L_{\mathbb{Q}} \cong \mathbb{Q}^r \oplus \left(\mathbb{Q}^-\right)^s \oplus (A_{n-1})^t_{\mathbb{Q}} \qquad (s + t \leq 2)$$

sign representation of $\mathcal{S}_n$

Example: $\mathcal{S}_n$-lattices

In all cases, $\mathbb{Z}[L]^{\mathcal{S}_n}$ is indeed CM, with the possible exception of

$$L = A_{n-1}^2$$

This case reduces to

Problem (open for $p \leq n/2$)

Are the “vector invariants" $\mathbb{F}_p[x_1, \ldots, x_n, y_1, \ldots, y_n]^{\mathcal{S}_n}$ CM?

Summary

Let L be a $\mathcal{G}$-lattice, where $\mathcal{G}$ is a finite group.