Ring theoretic methods in the representation theory of Hopf algebras

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Thank you

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A MORITA CONTEXT RELATED TO FINITE AUTOMORPHISM GROUPS OF RINGS

MIRIAM COHEN

Let R be a semiprime ring, G a finite group of automorphisms of R and R^G the fixed ring. We investigate the associated Morita context $(R^G, R, R, R*G)$, where $R*G$ is the skew group ring. We then apply these results to two situations: (1)G is X-outer (2)R is $|G|$-torsion free.

0. **Introduction and preliminaries.** Let R be a ring, G a finite group of automorphisms of R and $R^G = \{x \in R \mid x^g = x$ for all $g \in G\}$. There has been considerable interest in the past years in studying connections between R^G and R. The two major ways to approach the subject were the direct approach, and via the skew group ring $R*G$ which we denote by S. In this paper we investigate a third way which was used in the commutative case by Chase, Harrison and Rosenberg [5], and was suggested to us by S.A. Amitsur.

Overview

(I) **Ring theory**:

- Background on Frobenius algebras

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- Separability locus

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- Background on Frobenius algebras
- Separability locus
- Characters

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- Characters
- Integrals

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(II) **Hopf algebras**:

- Frobenius Hopf algebras

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- Frobenius Hopf algebras
- Grothendieck rings

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- Frobenius Hopf algebras
- Grothendieck rings
- A theorem of Shenglin Zhu

Overview

(III) **More ring theory**:

- Projectives over Frobenius algebras

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- The Cartan-Brauer triangle

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(IV) **More Hopf algebras**:

- Bialgebra cohomology

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(III) **More ring theory**:

- Projectives over Frobenius algebras
- The Cartan-Brauer triangle

(IV) **More Hopf algebras**:

- Bialgebra cohomology
- Some problems (if time)

Reference

- *“Some applications of Frobenius algebras to Hopf algebras”* preprint covering parts (I) and (II) & . . .

Article & **pdf file of this talk** available on my web page:

http://math.temple.edu/~lorenz/

Frobenius algebras

Notation

For the remainder of this talk,

R	denotes a commutative ring
A	is an associative R-algebra that is fin. gen. projective (“finite”) over R

Definitions

Put $A^* = \mathrm{Hom}_R(A, R)$; this is an (A, A)-bimodule via the (A, A)-bimodule structure on A:

$$(afb)(x) = f(bxa) \qquad (a, b, x \in A, f \in A^*) \ .$$

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The algebra A is called Frobenius if it has the following equivalent properties:

- $A \cong A^*$ as right A-modules;
- there exists a nonsingular R-bilinear form $\beta\colon A \times A \to R$ that is associative: $\beta(ab, c) = \beta(a, bc)$;
- $A \cong A^*$ as left A-modules.

later

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Similarly, A is called symmetric if it has the following equivalent properties

- $A \cong A^*$ as (A, A)-bimodules;
- there exists a nonsingular associative R-bilinear form $\beta\colon A \times A \to R$ that is *symmetric*: $\beta(a, b) = \beta(b, a)$.

Nonsingularity

The R-bilinear form $\beta\colon A \times A \to R$ is said to be nonsingular if it satisfies the following equivalent conditions:

- the map $A \to A^*$, $a \mapsto \beta(\,.\,, a)$, is an isomorphism;
- there exist dual bases $\{x_i\}_1^n \subseteq A$ and $\{y_i\}_1^n \subseteq A$ satisfying

$$a = \sum_i \beta(a, y_i) x_i \quad \text{for all } a \in A.$$

These conditions are actually left-right symmetric, even if β is not symmetric.

Examples and remarks

(a) Symmetry and the Frobenius property are stable under base change $R \to R'$.

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faithful & separable $/R \quad \Rightarrow \quad$ symmetric $/R$

Examples and remarks

(a) Symmetry and the Frobenius property are stable under base change $R \to R'$.

(b) **Theorem of Endo and Watanabe** (1967):

$$\text{faithful \& separable } /R \quad \Rightarrow \quad \text{symmetric } /R$$

(c) Each (A, A)-bimodule isomorphism $A \xrightarrow{\sim} A^*$ restricts to

center

$f(ab) = f(ba)$

$$\mathcal{Z}(A) \xrightarrow{\sim} A^*_{\text{trace}}$$

Separability locus

Goal: For a given Frobenius R-algebra A, determine

$$\{\mathfrak{p} \in \operatorname{Spec} R \mid A \otimes_R Q(R/\mathfrak{p}) \text{ is separable}\}$$

$\iff$ $A \otimes_R R_{\mathfrak{p}}$ is separable

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We use techniques and results from

Donald G. Higman, *On orders in separable algebras*, Canad. J. Math. **7** (1955), 509–515

Separability locus

For a fixed nonsingular associative R-bilinear form $\beta\colon A \times A \to R$, define the Casimir operator

$$c_\beta\colon A \to \mathcal{Z}(A)\,, \quad a \mapsto \sum_i y_i a x_i$$

where $\{x_i\}_1^n, \{y_i\}_1^n \subseteq A$ are dual bases for β.

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One checks:

- c is independent of the choice of dual bases;
- $c(A)$ is an ideal of $\mathcal{Z}(A)$ which is independent of the choice of β ("Casimir ideal").

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Of particular importance will be the Casimir element

for A symmetric

$$z_{\beta} = c(1) = \sum_i y_i x_i = \sum_i x_i y_i \in \mathcal{Z}(A)$$

Separability locus

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Thm 1 $A \otimes_R Q(R/\mathfrak{p})$ *is separable* $\iff$ $\mathfrak{p} \not\supseteq c(A) \cap R$.

Characters

M a left A-module, f.g. projective $/R$

Trace $\mathrm{Tr}\colon \mathrm{End}_R(M) \cong M \otimes_R M^* \xrightarrow{\text{eval.}} R$

Rank $\mathrm{rank}_R M = \mathrm{Tr}(1_M) \in R$ (Hattori-Stallings)

Character $\chi_M\colon A \to R\,, \quad a \mapsto \mathrm{Tr}(a_M)$

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Character $\chi_M \colon A \to R\,, \quad a \mapsto \mathrm{Tr}(a_M)$

Central character $\omega_M \colon \mathcal{Z}(A) \to R\,, \quad a_M = \omega_M(a) 1_M$

$$\Downarrow$$

$$\chi_M(a) = \omega_M(a)\, \mathrm{rank}_R\, M$$

Characters

(A, β) a symmetric R-algebra, with dual bases $\{x_i\}, \{y_i\} \subseteq A$
M a left A-module, f.g. projective $/R$,

$$\rightsquigarrow \quad \begin{array}{ccc} A^*_{\mathrm{trace}} & \underset{\beta}{\overset{\sim}{\longrightarrow}} & \mathcal{Z}(A) \\ \Psi & & \Psi \\ \chi_M & \longleftrightarrow & z_\beta(M) = \sum_i \chi_M(x_i) y_i \end{array}$$

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Example: $M = A$ yields the regular character

$$\chi_{\text{reg}} := \chi_A \leftrightarrow z_\beta = \sum_i y_i x_i = \sum_i x_i y_i$$

Casimir element

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If $\operatorname{End}_A(M) \cong R$ then define the **index**

$$[A : M]_\beta := \omega_M(z(M)) \in R$$

Characters

Theorem 2 *Assume that* A *is separable and that* M *is cyclic with* $\mathrm{End}_A(M) \cong R$*. Then:*

(a) $[A : M]$ *is invertible in* R*.*

(b) $e(M) := [A : M]^{-1} z(M) \in \mathcal{Z}(A)$ *is an idempotent such that* $e(M)_M = 1_M$ *and* $x e(M) = \omega_M(x) e(M)$ $(x \in \mathcal{Z}(A))$*.*

(c) $\chi_{\mathrm{reg}}\, e(M) = (\mathrm{rank}_R\, M)\chi_M$

Integrals

Let (A, β) be a Frobenius algebra, **augmented** by

$$\varepsilon \colon A \to R \ .$$

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Let (A, β) be a Frobenius algebra, **augmented** by

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Define $\Lambda_\beta \in A$ by $\beta(\Lambda_\beta, \, . \,) = \varepsilon$. With dual bases $\{x_i\}, \{y_i\}$ for β,

$$\Lambda_\beta = \sum_i \varepsilon(y_i) x_i \; .$$

Integrals

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The **right integrals** in A are given by

$$\int_A^r := \{t \in A \mid ta = \varepsilon(a)t\ \forall a \in A\} = R\Lambda_\beta \cong R$$

Integrals

Let (A, β) be a Frobenius algebra, **augmented** by

$$\varepsilon\colon A \to R\ .$$

Similarly, one defines the R-module $\int_A^l$ of **left integrals** and shows

$$\int_A^l = R\Lambda'_\beta \cong R \qquad \text{with } \Lambda'_\beta = \sum_i \varepsilon(x_i) y_i$$

Integrals

Let (A, β) be a Frobenius algebra, **augmented** by

$$\varepsilon\colon A \to R\ .$$

If A is symmetric then $\Lambda_\beta = \Lambda'_\beta$ and

$$\boxed{\int_A^r = \int_A^l =: \int_A}$$

Hopf algebras

Notations

Throughout, we consider Hopf algebras that are **finite** $/R$:

$$H = (H, u, m, \varepsilon, \Delta, \mathcal{S})$$

The bimodule operations for ${}_H H^*{}_H$ and ${}_{H^*} H_{H^*}$ will be denoted by $\rightharpoonup$ and $\leftharpoonup$:

$$\langle a \rightharpoonup f \leftharpoonup b, c\rangle = \langle f, bca\rangle \qquad (a, b, c \in H, f \in H^*) \ ,$$
$$\langle e, f \rightharpoonup a \leftharpoonup g\rangle = \langle gef, a\rangle \qquad (e, f, g \in H^*, a \in H) \, .$$

Here, $\langle\, . \, , \, . \,\rangle \colon H^* \times H \longrightarrow R$ denotes the evaluation pairing.

Theorem 3 (Larson-Sweedler ’69, Pareigis ’71, Oberst-Schneider ’73)

(a) *The antipode* $\mathcal{S}$ *is bijective. Consequently,* $\int_H^l = \mathcal{S}(\int_H^r)$.

(b) H *is Frobenius if and only if* $\int_H^r \cong R$. *This is automatic if* $\operatorname{Pic} R = 1$. *Furthermore, if* H *is Frobenius then so is* H^*.

(c) *Assume* H *is Frobenius. Then* H *is symmetric iff*

(i) H *is unimodular (i.e.,* $\int_H^l = \int_H^r$*), and*

(ii) $\mathcal{S}^2$ *is an inner automorphism of* H.

Frobenius Hopf algebras

Assume H Frobenius and fix a generator $\Lambda \in \int_H^r$.

$$\rightsquigarrow$$

Frobenius Hopf algebras

Assume H Frobenius and fix a generator $\Lambda \in \int_H^r$.

$$\rightsquigarrow$$

bilinear form: There is a unique $\lambda \in \int_{H^*}^l$ with $\langle \lambda, \Lambda \rangle = 1$.

$$\boxed{\beta(a,b) = \langle \lambda, ab \rangle}$$

dual bases: $\{x_i\} = \{\Lambda_2\}$, $\{y_i\} = \{\mathcal{S}(\Lambda_1)\}$ $(\Delta(\Lambda) = \sum \Lambda_1 \otimes_R \Lambda_2)$

Frobenius Hopf algebras

Assume H Frobenius and fix a generator $\Lambda \in \int_H^r$.

$$\rightsquigarrow$$

The **Casimir operator** is given by the right adjoint action of Λ on H:

$$c\colon H \longrightarrow \mathcal{Z}(H)\ , \quad a \mapsto \sum \mathcal{S}(\Lambda_1)a\Lambda_2$$

Casimir element:

$$\boxed{z = c(1) = \langle \varepsilon, \Lambda \rangle \in R}$$

Theorem 1 gives a classical result due to Larson and Sweedler:

Corollary 1 *The separability locus of a Frobenius Hopf algebra H over R is*

$$\operatorname{Spec} R \setminus V(\langle \varepsilon, \textstyle\int_H^r \rangle)$$

Grothendieck rings

In this part:

$\Bbbk$	is an alg. closed field, $\operatorname{char} \Bbbk = 0$
H	is a semisimple Hopf algebra $/\Bbbk$
$\operatorname{Irr} H$	is a full set of irreducible H-modules

The **Grothendieck ring** is

the (tensor) category of f.g. left H-modules

$$G_0(H) = K_0(H\text{-mod}) = \bigoplus_{V \in \operatorname{Irr} H} \mathbb{Z}[V]$$

Grothendieck rings

The Grothendieck ring $G_0(H)$ is a symmetric $\mathbb{Z}$-algebra.

bilinear form: $\beta([V],[W]) = \dim_{\Bbbk} \operatorname{Hom}_H(V, W^*)$

$\Bbbk$-linear dual

dual bases: $\{[V] \mid V \in \operatorname{Irr} H\}$, $\{[V^*] \mid V \in \operatorname{Irr} H\}$

Grothendieck rings

The Grothendieck ring $G_0(H)$ is a symmetric $\mathbb{Z}$-algebra.

Casimir element:

$$z = \sum_{V \in \operatorname{Irr} H} [V^*][V] = [H_{\mathrm{ad}}] \ ,$$

the class of $\mathrm{ad} = \mathrm{ad}_l \colon H \longrightarrow \mathrm{End}_{\Bbbk}(H)$, $\mathrm{ad}(h)(k) = \sum h_1 k \mathcal{S}(h_2)$.

Application: separability locus of $G_0(H)$

By Thm 1, the issue is to determine the ideal $c(G_0(H)) \cap \mathbb{Z}$ for the **Casimir operator**

$$c\colon G_0(H) \to \mathcal{Z}(G_0(H)), \quad [M] \mapsto \sum_{V \in \operatorname{Irr} H} [V^*][M][V]\ .$$

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Here is what I know . . .

Application: separability locus of $G_0(H)$

Theorem 4

(a) *If* p *divides* $\dim_{\Bbbk} H$ *then* $G_0(H) \otimes_{\mathbb{Z}} \mathbb{F}_p$ *is not semisimple.*

(b) $G_0(H) \otimes_{\mathbb{Z}} \mathbb{F}_p$ *is semisimple for* $p = 0$ *and all* $p > \dim_{\Bbbk} H$*.*

(c) *If* $G_0(H)$ *is commutative then* $G_0(H) \otimes_{\mathbb{Z}} \mathbb{F}_p$ *is semisimple if and only if* p *does not divide* $\dim_{\Bbbk} H$*.*

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Part (a) follows from the augmentation $\dim = \dim_{\Bbbk} \colon G_0(H) \to \mathbb{Z}$

$$(\dim_{\Bbbk} H) = (\dim z) = \dim c(G_0(H)) \supseteq c(G_0(H)) \cap \mathbb{Z} \ .$$

Application: separability locus of $G_0(H)$

Theorem 4

(a) *If p divides $\dim_{\Bbbk} H$ then $G_0(H) \otimes_{\mathbb{Z}} \mathbb{F}_p$ is not semisimple.*

(b) *$G_0(H) \otimes_{\mathbb{Z}} \mathbb{F}_p$ is semisimple for $p = 0$ and all $p > \dim_{\Bbbk} H$.*

(c) *If $G_0(H)$ is commutative then $G_0(H) \otimes_{\mathbb{Z}} \mathbb{F}_p$ is semisimple if and only if p does not divide $\dim_{\Bbbk} H$.*

For (b) and (c), I use the norm of the adjoint class $z = c(1) = [H_{\mathrm{ad}}]$.

(class equation, Sommerhäuser)

The character map

The *character map*

$$\chi : G_0(H) \longrightarrow H^* , \qquad [V] \mapsto \chi_V$$

is a ring embedding.

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is a ring embedding. In fact, χ yields an isomorphism

$$G_0(H) \otimes_{\mathbb{Z}} \Bbbk \cong H^*_{\text{trace}} = \bigoplus_{V \in \operatorname{Irr} H} \Bbbk\chi_V \subseteq H^*$$

This is a semisimple $\Bbbk$-algebra (Thm 4) with $\mathbb{Z}$-form $G_0(H)$.

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is a ring embedding. It respects

- augmentations:

$$\begin{array}{ccc} G_0(H) & \overset{\chi}{\hookrightarrow} & H^* \\ {\scriptstyle \dim}\downarrow & & \downarrow{\scriptstyle u^*} \\ \mathbb{Z} & \hookrightarrow & \Bbbk \end{array}$$

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$$\int_{G_0(H)} = \mathbb{Z}\,[H_{\text{reg}}] \hookrightarrow \Bbbk\chi_{\text{reg}} = \int_{H^*}$$

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- augmentations: $$\begin{array}{ccc} G_0(H) & \overset{\chi}{\hookrightarrow} & H^* \\ {\scriptstyle \dim}\downarrow & & \downarrow{\scriptstyle u^*} \\ \mathbb{Z} & \hookrightarrow & \Bbbk \end{array}$$
- integrals: $$\textstyle\int_{G_0(H)} = \mathbb{Z}\,[H_{\mathrm{reg}}] \hookrightarrow \Bbbk\chi_{\mathrm{reg}} = \int_{H^*}$$
- involutions: $$\chi_{V^*} = \mathcal{S}^*(\chi_V)$$

Application: a theorem of Shenglin Zhu

Theorem 5 (S. Zhu '93)

If $V \in \operatorname{Irr} H$ *satisfies* $\chi_V \in \mathcal{Z}(H^*)$ *then* $\dim_{\Bbbk} V$ *divides* $\dim_{\Bbbk} H$*.*

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Sketch of proof (an application of Thm 2):

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Sketch of proof (an application of Thm 2): Put $\Lambda = \chi_{H^*_{\mathrm{reg}}} \in \int_H$ and consider

$$\boldsymbol{b}\colon {}_{H^*}{H^*}_{H^*} \xrightarrow{\sim} {}_{H^*}H_{H^*}\ , \qquad f \mapsto (f \rightharpoonup \Lambda = \Lambda \leftharpoonup f)$$

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Thm 2(c) gives $\boldsymbol{b}(\chi_{V^*}) = \frac{\dim_{\Bbbk} H}{\dim_{\Bbbk} V} e(V)$; so we need to show:

$$\boldsymbol{b}(\chi_{V^*}) \text{ is integral } /\mathbb{Z}$$

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$\boldsymbol{b}(\chi_{V^*})$ is integral $/\mathbb{Z}$

where $\boldsymbol{b}\colon {}_{H^*}H^*{}_{H^*} \xrightarrow{\sim} {}_{H^*}H_{H^*}$, $f \mapsto (f \rightharpoonup \Lambda = \Lambda \leftharpoonup f)$.

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where $\boldsymbol{b}\colon {}_{H^*}H^*{}_{H^*} \xrightarrow{\sim} {}_{H^*}H_{H^*}$, $f \mapsto (f \rightharpoonup \Lambda = \Lambda \leftharpoonup f)$. Now,

$$\chi_{V^*} \underset{\text{hypo.}}{\in} \mathcal{Z}(H^*)^{\text{alg. int.}} = \bigoplus_{M \in \operatorname{Irr} H^*} \Bbbk^{\text{alg. int.}} e(M) \xrightarrow[\text{Thm 2(c)}]{\boldsymbol{b}} \chi(G_0(H^*))\Bbbk^{\text{alg. int.}} \subseteq H$$

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Finally, all elements of $\chi(G_0(H^*))\Bbbk^{\text{alg. int.}}$ are integral $/\mathbb{Z}$. ■

Ring theory again . . .

Notations

In this part,

$(F, R, \Bbbk)$	consists of a complete d.v.r. R with residue field $\Bbbk = R/\mathfrak{m}$ and quotient field $F = Q(R)$
A	is a Frobenius R-algebra, finite $/R$
$\overline{}$	denotes "reduction mod $\mathfrak{m}$": $\overline{(\,.\,)} = \Bbbk \otimes_R (\,.\,)$
$\cdot_{(F)}$	denotes base change $R \to F$: $\cdot_{(F)} = F \otimes_R \,\cdot$

Projectives over Frobenius algebras

The following transports standard standard methods and results for finite group algebras to the setting of general Frobenius algebras.

Reference: J.-P. Serre, *Linear Representations of Finite Groups*, Chapters 14 and 15

The framework I use is that of “Frobenius extensions”.

Proposition 5

(a) *Let V be a f.g. A-module. Then:*

$$V \text{ is projective} \iff V\big|_R \text{ and } \overline{V} \text{ are projective}$$

(b) *Let P and P' be f.g. projective A-modules. Then:*

$$P \cong P' \iff \overline{P} \cong \overline{P'}$$

(c) *For every f.g. projective $\overline{A}$-module Q, there exists a f.g. projective A-module P such that $Q \cong \overline{P}$.*

Projectives over Frobenius algebras

Now consider the Grothendieck group

the category of f.g. projective left A-modules

$$K_0(A) = K_0(A\text{-proj})$$

This is a partially ordered abelian group with positive cone

$$K_0^+(A) = \{[P] \mid P \text{ in } A\text{-proj}\}$$

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Corollary 6

(a) $[P] \mapsto [\overline{P} = P/\mathfrak{m}P]$ *gives an isomorphism of p. ordered abelian groups,* $K_0(A) \xrightarrow{\sim} K_0(\overline{A})$.

(b) *For* P *and* Q *in* A-proj, $[P] = [Q]$ *in* $K_0(A)$ *iff* $P \cong Q$.

The Cartan-Brauer triangle

$$
\begin{array}{ccc}
K_0(\overline{A}) & \xrightarrow{\quad c \quad} & G_0(\overline{A}) \\
 & {\scriptstyle e}\searrow \quad \circlearrowleft \quad \nearrow{\scriptstyle d} & \\
 & G_0(A_{(F)}) &
\end{array}
$$

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- c is the Cartan map, from $\overline{A}$-proj $\hookrightarrow$ $\overline{A}$-mod
- $e = \left(K_0(\overline{A}) \underset{\text{Cor. 6}}{\overset{\sim}{\to}} K_0(A) \xrightarrow[F \otimes_R (\,.\,)]{} K_0(A_{(F)}) \xrightarrow[\text{Cartan}]{} G_0(A_{(F)}) \right)$

The Cartan-Brauer triangle

$$\begin{array}{ccc} K_0(\overline{A}) & \xrightarrow{\quad c \quad} & G_0(\overline{A}) \\ & {\scriptstyle e}\searrow \quad \circlearrowleft \quad \nearrow{\scriptstyle d} & \\ & G_0(A_{(F)}) & \end{array}$$

- c is the Cartan map, from $\overline{A}$-proj $\hookrightarrow$ $\overline{A}$-mod
- $e = \left(K_0(\overline{A}) \underset{\text{Cor. 6}}{\xrightarrow{\sim}} K_0(A) \underset{F \otimes_R (\,.\,)}{\longrightarrow} K_0(A_{(F)}) \underset{\text{Cartan}}{\longrightarrow} G_0(A_{(F)}) \right)$
- d is the decomposition map: given V in $A_{(F)}$-mod choose L in A-mod such that $L \subseteq V$, $V = FL$ and put $d([V]) := [\overline{L}]$; this is independent of the choice of L. (R. G. Swan)

The Cartan-Brauer triangle

Proposition 7 *Assume that $\overline{A}$ is semisimple. Then $A_{(F)}$ is semisimple as well. Moreover:*

(a) *The maps d and e are order preserving isomorphisms that are inverse to each other:*

$$G_0(\overline{A}) \underset{d}{\overset{e}{\rightleftarrows}} G_0(A_{(F)})$$

(b) *In particular, d and e yield a dimension preserving one-to-one correspondence* $\operatorname{Irr}\overline{A} \leftrightarrow \operatorname{Irr} A_{(F)}$.

(c) *$A_{(F)}$ is F-split if and only if $\overline{A}$ is $\Bbbk$-split.*

Back to Hopf algebras . . .

Bialgebra cohomology (Gerstenhaber-Schack)

Notation:

$\mathbb{k}$ is a field

B a $\mathbb{k}$-bialgebra

Bialgebra cohomology (Gerstenhaber-Schack)

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Define a bicomplex $\mathbf{B} = (\mathbf{B}^{p,q})_{p,q \geq 0}$ with $\mathbf{B}^{p,q} = \mathrm{Hom}_{\mathbb{k}}(B^{\otimes p}, B^{\otimes q})$ and with $\delta^{p,q}_{\text{Hochschild}} : \mathbf{B}^{p,q} \to \mathbf{B}^{p+1,q}$ and $\delta^{p,q}_{\text{Cartier}} : \mathbf{B}^{p,q} \to \mathbf{B}^{p,q+1}$.

$$H^{\bullet}_{\mathrm{GS}}(B) := H^{\bullet}(\mathrm{Tot}\,\mathbf{B})$$

Bialgebra cohomology (Gerstenhaber-Schack)

For the purposes of studying bialgebra deformations, the following variant is most useful.

Let $\widehat{\mathrm{B}}$ denote the sub-bicomplex of B where the edge row and column are replaced by zeroes.

$$\widehat{H}^{\bullet}_{\mathrm{GS}}(B) := H^{\bullet+1}(\mathrm{Tot}\,\widehat{\mathbf{B}})$$

Bialgebra cohomology (Gerstenhaber-Schack)

Proposition 8 *Each of the following conditions implies that* $\widehat{H}^{\bullet}_{\mathrm{GS}}(B) = 0$*.*

(a) B *is separable as* $\Bbbk$*-algebra and commutative;*

(b) *the dual algebra* B^* *is separable and commutative;*

(c) B *is a bi-semisimple Hopf algebra.* *(D. Ştefan)*

e.g., $B = \Bbbk G$ a finite group algebra

Bialgebra cohomology (Gerstenhaber-Schack)

Proposition 9 (Etingof-Gelaki)

Let $(F, R, \Bbbk)$ be as before and let B be a finite-dimensional $\Bbbk$-bialgebra.

(a) *If $\widehat{H}^3_{\mathrm{GS}}(B) = 0$ then there exists an R-free R-bialgebra $\widetilde{B}$ such that $\widetilde{B} \otimes_R \Bbbk = B$. Moreover, if B is a Hopf algebra then so is each such $\widetilde{B}$.*

(b) *If $\widehat{H}^2_{\mathrm{GS}}(B) = 0$ then, up to isomorphism, there is at most one $\widetilde{B}$ as in (a).*

Problem: lifting to characteristic 0

Is it conceivable that all semisimple Hopf algebras over a field $\Bbbk$ of $\operatorname{char} p > 0$ lift to $\operatorname{char} 0$?

Problem: adjoint representation and Chevalley property

Notation:

$\Bbbk$ is an alg. closed field, $\operatorname{char} \Bbbk = p > 0$

G is a finite group