Algebraic group actions on noncommutative spectra

Special Session on Brauer Groups, Quadratic Forms, Algebraic Groups, and Lie Algebras — NCSU 04/04/2009

Martin Lorenz
Temple University, Philadelphia

Overview

- Background: enveloping algebras and quantized coordinate algebras

Overview

- Background: enveloping algebras and quantized coordinate algebras
- Tool: the Amitsur-Martindale ring of quotients

Overview

- Background: enveloping algebras and quantized coordinate algebras
- Tool: the Amitsur-Martindale ring of quotients
- Rational and primitive ideals

Overview

- Background: enveloping algebras and quantized coordinate algebras
- Tool: the Amitsur-Martindale ring of quotients
- Rational and primitive ideals
- Stratification of the prime spectrum (if time)

References

- *“Group actions and rational ideals”*,
  Algebra and Number Theory **2** (2008), 467-499

- *“Algebraic group actions on noncommutative spectra”*,
  Transformation Groups (to appear)

Both articles & the **pdf file of this talk** available on my web page:

http://math.temple.edu/~lorenz/

I will work / base field $\mathbb{k} = \overline{\mathbb{k}}$

Background

Enveloping algebras

Goal: For $R = U(\mathfrak{g})$, the enveloping algebra of a finite-dim'l Lie algebra $\mathfrak{g}$, describe

$$\operatorname{Prim} R = \{\text{primitive ideals of } R\}$$

kernels of irreducible (generally infinite-dimensional) representations $R \longrightarrow \operatorname{End}_{\Bbbk}(V)$

Jacques Dixmier (* 1924)

in Reims, Dec. 2008

- former secretary of Bourbaki
- Ph.D. advisor of A. Connes, M. Duflo, ...
- author of several influential monographs:

Les algèbres d'opérateurs dans l'espace hilbertien: algèbres de von Neumann, Gauthier-Villars, 1957

Les C^*-algèbres et leurs représentations, Gauthier-Villars, 1969

Algèbres enveloppantes, Gauthier-Villars, 1974 from *Algèbres enveloppantes*, 1974 :

10. On suppose que tr ad $x = 0$ pour tout $x \in \mathfrak{g}$. Est ce que $Z(\mathfrak{g}) \neq k$?

11. Soient $\mathfrak{k}$ un idéal de $\mathfrak{g}$, I un idéal primitif de $U(\mathfrak{g})$. Les propriétés suivantes sont-elles vraies : (*a*) il existe un idéal primitif de $U(\mathfrak{k})$ générique pour $U(\mathfrak{k}) \cap I$; (*b*) deux tels idéaux sont conjugués par le groupe adjoint algébrique de $\mathfrak{g}$; (*c*) soit L un tel idéal; il existe une représentation simple σ de $\mathfrak{k}$ de noyau L, et une représentation simple ρ de $\mathfrak{st}(\sigma, \mathfrak{g})$, telles que $\rho \mid \mathfrak{k}$ soit un multiple de σ et que $\text{ind}(\rho, \mathfrak{g})$ soit simple de noyau I. *Cf.* 4.5.9, 5.4.3, 5.4.4, 5.6.5.

Dixmier's Problem 11

- Problem 11 for $\mathfrak{k}$ solvable, $\operatorname{char} \mathbb{k} = 0$

Dixmier, *Sur les idéaux génériques dans les algèbres enveloppantes*, Bull. Sci. Math. (2) **96** (1972), 17–26. $\rightsquigarrow$ existence: (a)

Borho, Gabriel, Rentschler, *Primideale in Einhüllenden auflösbarer Lie-Algebren*, Springer Lect. Notes in Math. 357 (1973). $\rightsquigarrow$ uniqueness: (b)

- for noetherian or Goldie rings R / $\operatorname{char} \mathbb{k} = 0$:
  Mœglin & Rentschler

Orbites d'un groupe algébrique dans l'espace des idéaux rationnels d'une algèbre enveloppante, Bull. Soc. Math. France **109** (1981), 403–426.

Sur la classification des idéaux primitifs des algèbres enveloppantes, Bull. Soc. Math. France **112** (1984), 3–40.

Sous-corps commutatifs ad-stables des anneaux de fractions des quotients des algèbres enveloppantes; espaces homogènes et induction de Mackey, J. Funct. Anal. **69** (1986), 307–396.

Idéaux G-rationnels, rang de Goldie, preprint, 1986.

Dixmier's Problem 11

- Problem 11 for $\mathfrak{k}$ solvable, $\operatorname{char} \mathbb{k} = 0$

Dixmier, *Sur les idéaux génériques dans les algèbres enveloppantes*, Bull. Sci. Math. (2) **96** (1972), 17–26. $\rightsquigarrow$ existence: (a)

Borho, Gabriel, Rentschler, *Primideale in Einhüllenden auflösbarer Lie-Algebren*, Springer Lect. Notes in Math. 357 (1973). $\rightsquigarrow$ uniqueness: (b)

- for noetherian or Goldie rings R / $\operatorname{char} \mathbb{k} = 0$:
  Mœglin & Rentschler

Orbites d'un groupe algébrique dans l'espace des idéaux rationnels d'une algèbre enveloppante, Bull. Soc. Math. France **109** (1981), 403–426.

Sur la classification des idéaux primitifs des algèbres enveloppantes, Bull. Soc. Math. France **112** (1984), 3–40.

Sous-corps commutatifs ad-stables des anneaux de fractions des quotients des algèbres enveloppantes; espaces homogènes et induction de Mackey, J. Funct. Anal. **69** (1986), 307–396.

Idéaux G-rationnels, rang de Goldie, preprint, 1986.

Dixmier's Problem 11

- under weaker Goldie hypotheses / $\mathrm{char}\,\Bbbk$ arbitrary:
  N. Vonessen

Actions of algebraic groups on the spectrum of rational ideals,
J. Algebra **182** (1996), 383–400.

Actions of algebraic groups on the spectrum of rational ideals. II,
J. Algebra **208** (1998), 216–261.

Quantum groups

Goal: For $R = \mathcal{O}_q(\mathbb{k}^n), \mathcal{O}_q(\mathrm{M}_n), \mathcal{O}_q(G) \ldots$ a quantized coordinate ring, describe

$$\operatorname{Spec} R = \{\text{prime ideals of } R\} \supseteq \operatorname{Prim} R$$

Quantum groups

Goal: For $R = \mathcal{O}_q(\Bbbk^n)$, $\mathcal{O}_q(\mathrm{M}_n)$, $\mathcal{O}_q(G) \ldots$ a quantized coordinate ring, describe

$$\boxed{\operatorname{Spec} R = \{\text{prime ideals of } R\} \supseteq \operatorname{Prim} R}$$

Typically, some algebraic torus T acts rationally by $\Bbbk$-algebra automorphisms on R; so have

$$\begin{array}{ccc} \operatorname{Spec} R & \longrightarrow & \operatorname{Spec}^T R = \{T\text{-stable primes of } R\} \\ P & \longmapsto & P:T = \bigcap_{g \in T} g.P \end{array}$$

$\rightsquigarrow$ T**-stratification of** $\operatorname{Spec} R$

(Goodearl & Letzter, . . . ; see the monograph by Brown & Goodearl)

$$\operatorname{Spec} R = \bigsqcup_{I \in \operatorname{Spec}^T R} \operatorname{Spec}_I R$$

$$\operatorname{Spec}_I R = \{P \in \operatorname{Spec} R \mid P : T = I\}$$

$\rightsquigarrow$ **T-stratification of $\operatorname{Spec} R$**

(Goodearl & Letzter, . . . ; see the monograph by Brown & Goodearl)

$$\operatorname{Spec} R = \bigsqcup_{I \in \operatorname{Spec}^T R} \operatorname{Spec}_I R$$

?

Notation

For the remainder of this talk,

R denotes an associative $\Bbbk$-algebra (with 1)

G is an affine algebraic $\Bbbk$-group acting rationally on R; so R is a $\Bbbk[G]$-comodule algebra.

Equivalently, we have a rational representation

$$\rho = \rho_R \colon G \longrightarrow \mathrm{Aut}_{\Bbbk\text{-alg}}(R) \subseteq \mathbf{GL}(R)$$

Tool: The Amitsur-Martindale ring of quotients

Original references

for prime rings R:

W. S. Martindale, III, *Prime rings satisfying a generalized polynomial identity*, J. Algebra **12** (1969), 576–584.

for general R:

S. A. Amitsur, *On rings of quotients*, Symposia Math., Vol. VIII, Academic Press, London, 1972, pp. 149–164.

The definition

In brief,

$$\mathrm{Q_r}(R) = \varinjlim_{I \in \mathscr{E}} \mathrm{Hom}(I_R, R_R)$$

where $\mathscr{E} = \mathscr{E}(R)$ is the filter of all $I \trianglelefteq R$ such that $\mathrm{l.\,ann}_R\, I = 0$.

The definition

In brief,

$$\mathrm{Q_r}(R) = \varinjlim_{I \in \mathscr{E}} \mathrm{Hom}(I_R, R_R)$$

where $\mathscr{E} = \mathscr{E}(R)$ is the filter of all $I \trianglelefteq R$ such that $\mathrm{l.\,ann}_R\, I = 0$.

Explicitly, elements of $\mathrm{Q_r}(R)$ are equivalence classes of right R-module maps

$$f \colon I_R \to R_R \quad (I \in \mathscr{E}) \; ,$$

the map f being equivalent to $f' \colon I'_R \to R_R$ $(I' \in \mathscr{E})$ if $f = f'$ on some $J \subseteq I \cap I'$, $J \in \mathscr{E}$.

The definition

In brief,

$$\mathrm{Q_r}(R) = \varinjlim_{I \in \mathscr{E}} \mathrm{Hom}(I_R, R_R)$$

where $\mathscr{E} = \mathscr{E}(R)$ is the filter of all $I \trianglelefteq R$ such that $\mathrm{l.\,ann}_R\, I = 0$.

Addition and multiplication of $\mathrm{Q_r}(R)$ come from addition and composition of R-module maps.

The definition

In brief,

$$\mathrm{Q_r}(R) = \varinjlim_{I \in \mathscr{E}} \mathrm{Hom}(I_R, R_R)$$

where $\mathscr{E} = \mathscr{E}(R)$ is the filter of all $I \trianglelefteq R$ such that $\mathrm{l.\,ann}_R\, I = 0$.

Sending $r \in R$ to the equivalence class of $\lambda_r \colon R \to R$, $x \mapsto rx$, yields an embedding of R as a subring of $\mathrm{Q_r}(R)$.

Extended centroid

Def^{n}: The extended centroid of R is defined by

$$\mathcal{C}(R) = \mathcal{Z}\,\mathrm{Q}_{\mathrm{r}}(R)$$

center

Fact: If R is prime then $\mathcal{C}(R)$ is a $\Bbbk$-field.

Examples

- If R is **simple**, or a finite product of simple rings, then

$$\mathrm{Q_r}(R) = R$$

Examples

- If R is **simple**, or a finite product of simple rings, then

$$\mathrm{Q_r}(R) = R$$

- For R **semiprime right Goldie**,

$$\mathrm{Q_r}(R) = \{q \in \mathrm{Q_{cl}}(R) \mid qI \subseteq R \text{ for some } I \lhd R \text{ with } \operatorname{ann}_R I = 0\}$$

classical quotient ring of R

In particular,

$$\mathcal{C}(R) = \mathcal{Z}\mathrm{Q_{cl}}(R)$$

Examples

- $R = U(\mathfrak{g})/I$ a semiprime image of the enveloping algebra of a finite-dimensional Lie algebra $\mathfrak{g}$. Then

$$\boxed{\mathrm{Q_r}(R) = \{ \text{ ad}\,\mathfrak{g}\text{-finite elements of } \mathrm{Q_{cl}}(R) \ \}}$$

Rational Ideals

Definition

Want: an intrinsic characterization of “primitivity”, ideally

in detail . . .

Definition

“coeur”

“Herz”

“heart”

“core”

Definition:

- Recall that $\mathcal{C}(R/P)$ is a $\Bbbk$-field for any $P \in \operatorname{Spec} R$. We call P rational if $\mathcal{C}(R/P) = \Bbbk$.

Definition

“coeur”

“Herz”

“heart”

“core”

Definition:

- Recall that $\mathcal{C}(R/P)$ is a $\Bbbk$-field for any $P \in \operatorname{Spec} R$. We call P rational if $\mathcal{C}(R/P) = \Bbbk$.

- Put $\operatorname{Rat} R = \{P \in \operatorname{Spec} R \mid P \text{ is rational }\}$; so

$$\operatorname{Rat} R \subseteq \operatorname{Spec} R$$

Connection with irreducible representations

Given an irreducible representation $f\colon R \to \mathrm{End}_{\Bbbk}(V)$, let $P = \mathrm{Ker}\, f$ be the corresponding primitive ideal of R.

Connection with irreducible representations

Given an irreducible representation $f\colon R \longrightarrow \mathrm{End}_{\Bbbk}(V)$, let $P = \mathrm{Ker}\, f$ be the corresponding primitive ideal of R.

- There **always** is an embedding of $\Bbbk$-fields

$$\mathcal{C}(R/P) \hookrightarrow \mathcal{Z}\left(\mathrm{End}_R(V)\right)$$

Connection with irreducible representations

Given an irreducible representation $f\colon R \to \mathrm{End}_{\Bbbk}(V)$, let $P = \mathrm{Ker}\, f$ be the corresponding primitive ideal of R.

- There **always** is an embedding of $\Bbbk$-fields

$$\mathcal{C}(R/P) \hookrightarrow \mathcal{Z}\left(\mathrm{End}_R(V)\right)$$

- **Typically**, $\mathrm{End}_R(V) = \Bbbk$ (“weak Nullstellensatz”); in this case

$$\boxed{\mathrm{Prim}\, R \subseteq \mathrm{Rat}\, R}$$

Examples

The weak Nullstellensatz holds for

- R any affine $\Bbbk$-algebra, $\Bbbk$ uncountable — Amitsur
- R an affine PI-algebra — Kaplansky
- $R = U(\mathfrak{g})$ — "Quillen's Lemma"
- $R = \Bbbk\Gamma$ with Γ polycyclic-by-finite — Hall, L.
- many quantum groups: $\mathcal{O}_q(\Bbbk^n)$, $\mathcal{O}_q(M_n(\Bbbk))$, $\mathcal{O}_q(G)$, ...

Examples

The weak Nullstellensatz holds for

- R any affine $\mathbb{k}$-algebra, $\mathbb{k}$ uncountable — Amitsur
- R an affine PI-algebra — Kaplansky
- $R = U(\mathfrak{g})$ — "Quillen's Lemma"
- $R = \mathbb{k}\Gamma$ with Γ polycyclic-by-finite — Hall, L.
- many quantum groups: $\mathcal{O}_q(\mathbb{k}^n)$, $\mathcal{O}_q(M_n(\mathbb{k}))$, $\mathcal{O}_q(G)$, ...

In fact, in all these examples except the first, it has been shown that, under mild restrictions on $\mathbb{k}$ or q,

$$\operatorname{Prim} R = \operatorname{Rat} R$$

The G-action on R induces actions on $\{$ideals of $R\}$, $\operatorname{Spec} R$, $\operatorname{Rat} R$, $\ldots$. As usual, $G\backslash ?$ denotes the orbit sets in question.

Definition: A proper G-stable ideal $I \lhd R$ is called G-prime if $A, B \underset{G\text{-stab}}{\trianglelefteq} R$, $AB \subseteq I \Rightarrow A \subseteq I$ or $B \subseteq I$. Put

$$G\text{-}\operatorname{Spec} R = \{G\text{-prime ideals of } R\}$$

Group action: G-prime and G-rational ideals

Prop$^{\text{n}}$ *The assignment* $\gamma\colon P \mapsto P:G = \bigcap_{g \in G} g.P$ *yields surjections*

$$\begin{array}{ccc} \operatorname{Spec} R & \xrightarrow{\ \gamma\ } & G\text{-}\operatorname{Spec} R \\ {\scriptstyle \textit{can.}}\downarrow & \nearrow & \\ G\backslash \operatorname{Spec} R & & \end{array}$$

Group action: G-prime and G-rational ideals

Definition: Let $I \in G\text{-}\operatorname{Spec} R$. The group G acts on $\mathcal{C}(R/I)$ and the invariants $\mathcal{C}(R/I)^G$ are a $\Bbbk$-field. We call I G-rational if $\mathcal{C}(R/I)^G = \Bbbk$. Put

$$G\text{-}\operatorname{Rat} R = \{G\text{-rational ideals of } R\}$$

Group action: G-prime and G-rational ideals

Definition: Let $I \in G\text{-}\operatorname{Spec} R$. The group G acts on $\mathcal{C}(R/I)$ and the invariants $\mathcal{C}(R/I)^G$ are a $\Bbbk$-field. We call I G-rational if $\mathcal{C}(R/I)^G = \Bbbk$. Put

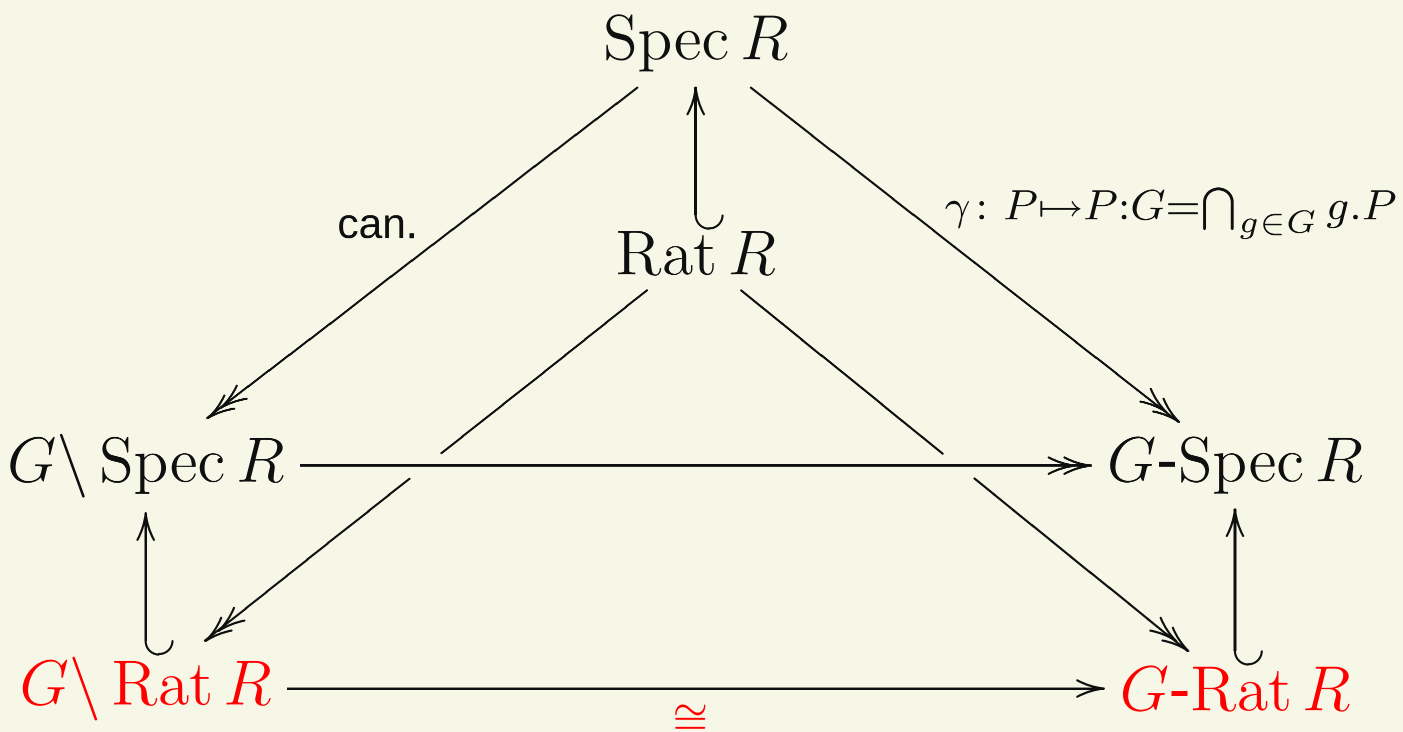
$$G\text{-}\operatorname{Rat} R = \{G\text{-rational ideals of } R\}$$

The following result solves Dixmier's Problem # 11 (a),(b) for arbitrary algebras.

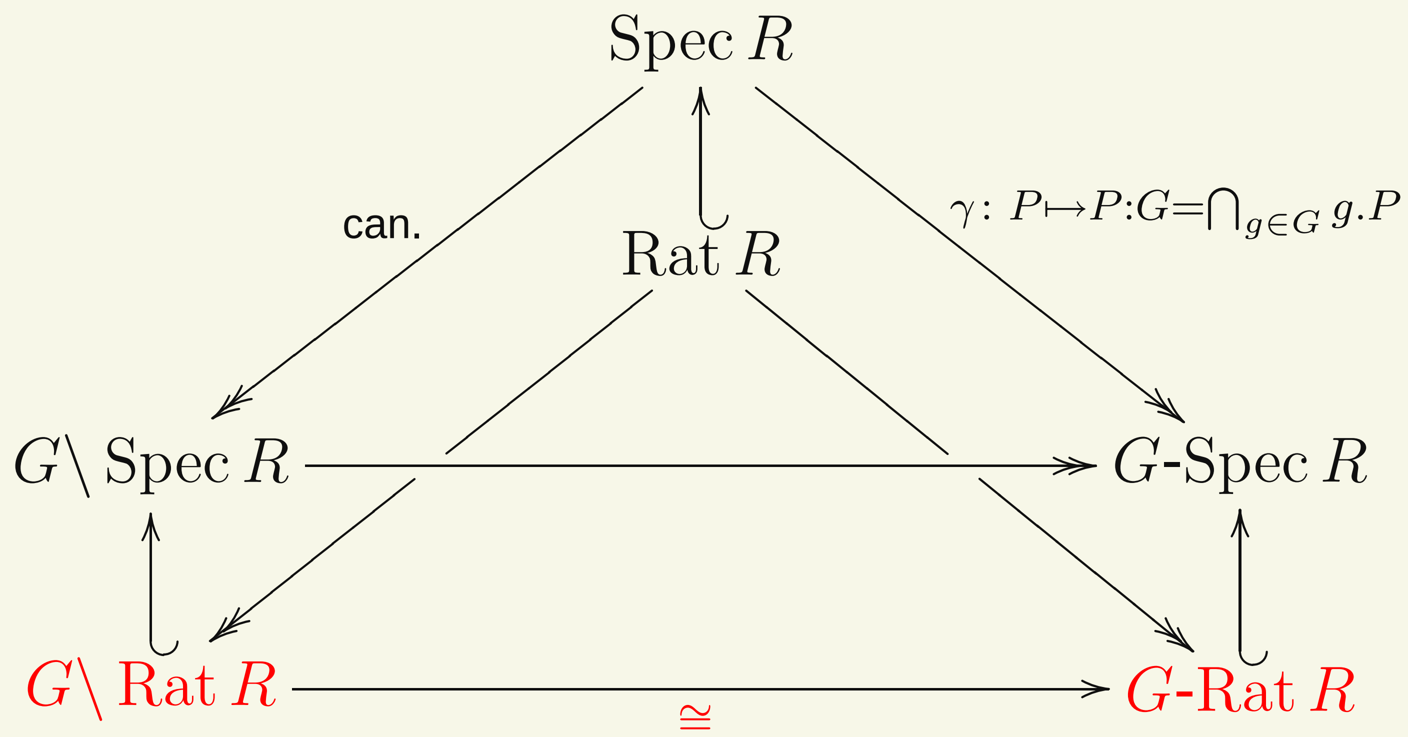
Theorem 1

$$\begin{array}{ccc} G\backslash \operatorname{Rat} R & \xrightarrow{\text{bij.}} & G\text{-}\operatorname{Rat} R \\ \Uparrow & & \Uparrow \\ G.P & \longmapsto & \bigcap_{g \in G} g.P \end{array}$$

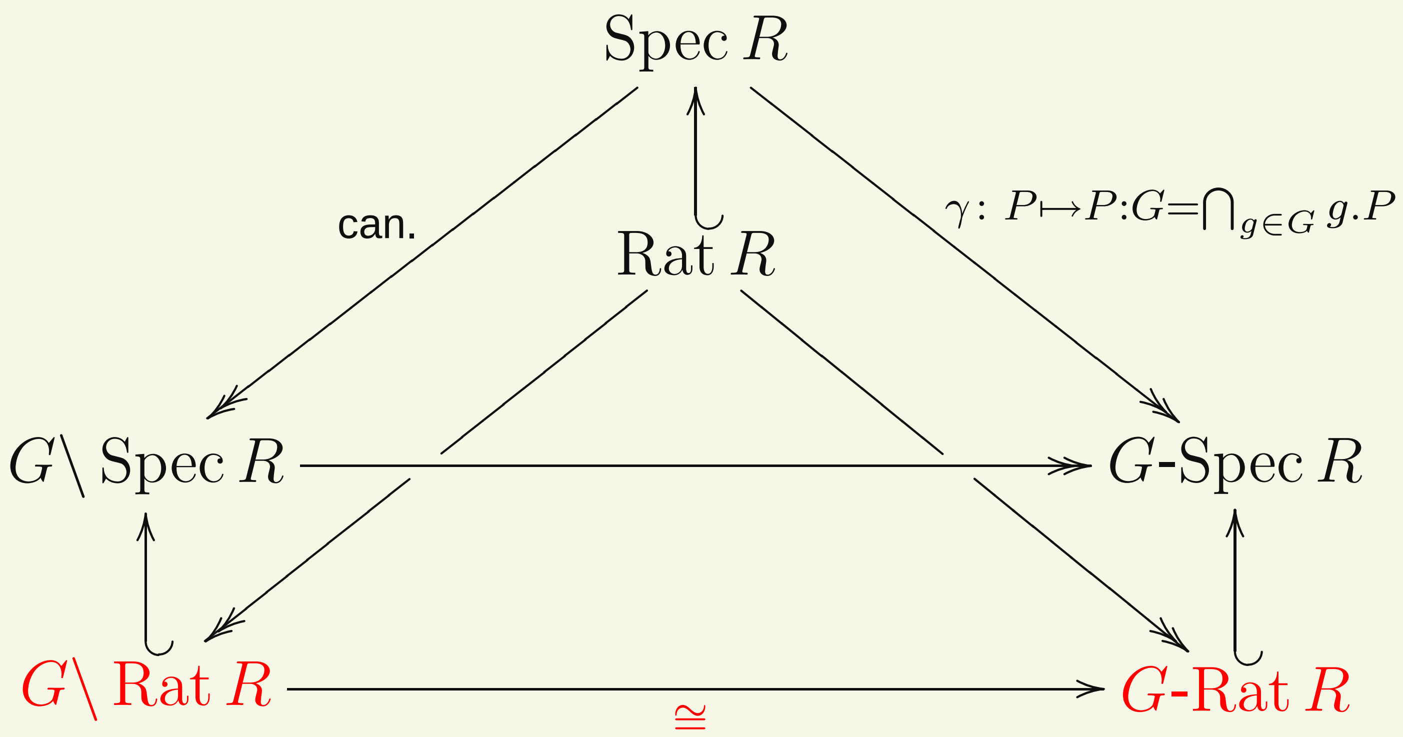
Noncommutative spectra



Noncommutative spectra



$\operatorname{Spec} R$ carries the Jacobson-Zariski topology: closed subsets are those of the form $\mathbf{V}(I) = \{P \in \operatorname{Spec} R \mid P \supseteq I\}$ where $I \subseteq R$.



$\twoheadrightarrow$ is a surjection whose target has the final topology,

$\hookrightarrow$ is an inclusion whose source has the induced topology, and

$\cong$ is a homeomorphism, from Thm 1

Local closedness

Recall: locally closed = open $\cap$ closed

Local closedness

Theorem 2 *If* $P \in \operatorname{Rat} R$ *then:*

$$\{P\} \text{ loc. cl. in } \operatorname{Spec} R \iff \{P{:}G\} \text{ loc. cl. in } G\text{-}\operatorname{Spec} R$$

Cor *If* $P \in \operatorname{Rat} R$ *is loc. closed in* $\operatorname{Spec} R$ *then the orbit* $G.P$ *is open in its closure in* $\operatorname{Rat} R$*.*

Local closedness

Theorem 2 *If* $P \in \operatorname{Rat} R$ *then:*

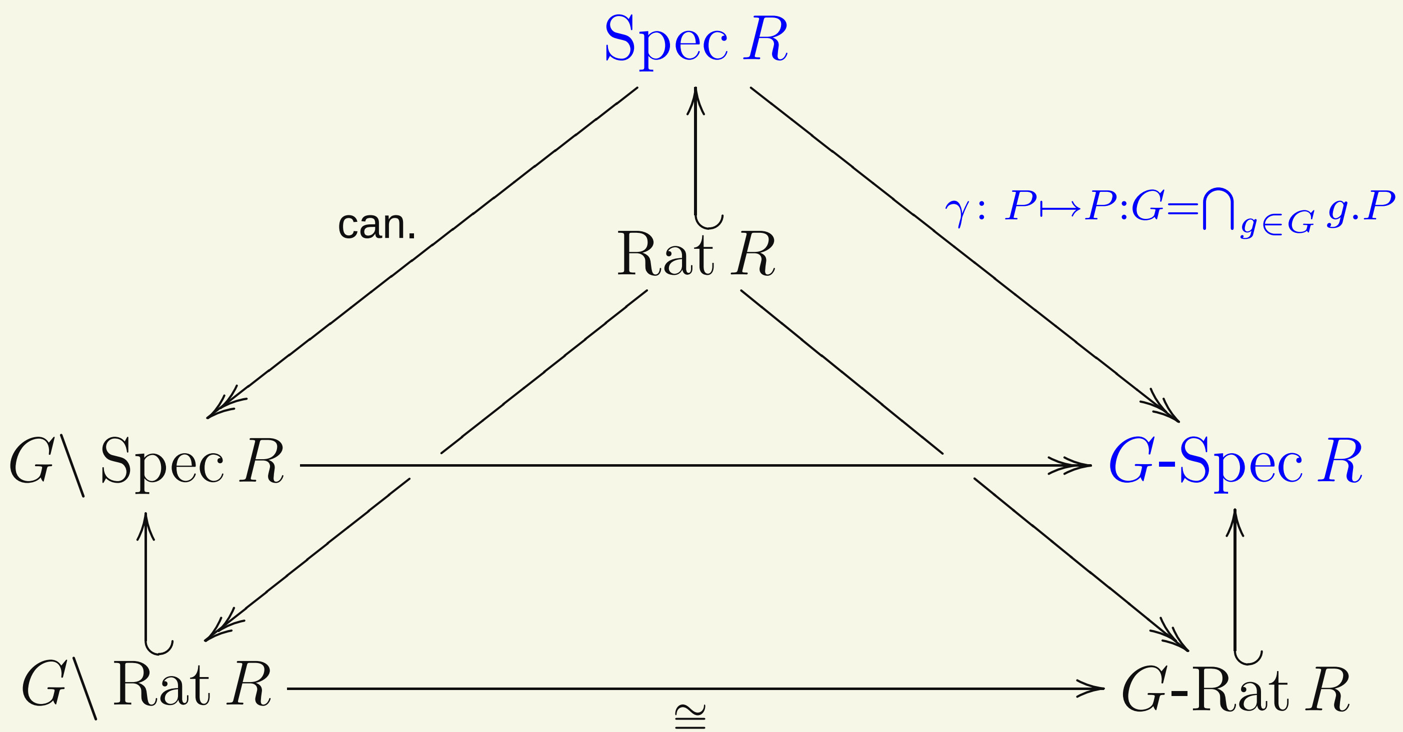
$$\{P\} \text{ loc. cl. in } \operatorname{Spec} R \iff \{P{:}G\} \text{ loc. cl. in } G\text{-}\operatorname{Spec} R$$

Cor *If* $P \in \operatorname{Rat} R$ *is loc. closed in* $\operatorname{Spec} R$ *then the orbit* $G.P$ *is open in its closure in* $\operatorname{Rat} R$.

Proof of Cor: $P{:}G \in G\text{-}\operatorname{Spec} R$ is locally closed by Theorem 2, and hence so is its preimage under $f\colon \operatorname{Rat} R \hookrightarrow \operatorname{Spec} R \xrightarrow{\gamma} G\text{-}\operatorname{Spec} R$. Finally, $f^{-1}(P{:}G) = G.P$ by Theorem 1.

Stratification of the prime spectrum

Goal



Next, we turn to the map γ ...

Goal

Recall: $\gamma\colon \operatorname{Spec} R \twoheadrightarrow G\text{-}\operatorname{Spec} R$ yields the G-stratification of $\operatorname{Spec} R$

$$\operatorname{Spec} R = \bigsqcup_{I \in G\text{-}\operatorname{Spec} R} \operatorname{Spec}_I R$$

Main goal: describe the G-strata

$$\boxed{\operatorname{Spec}_I R = \gamma^{-1}(I) = \{P \in \operatorname{Spec} R \mid P{:}G = I\}}$$

Goal

For simplicity, I assume G to be **connected**; so $\Bbbk[G]$ is a domain.
In particular,

$$G\text{-}\operatorname{Spec} R = \operatorname{Spec}^G R = \{G\text{-stable primes of } R\}$$

The rings T_I

For a given $I \in G\text{-}\operatorname{Spec} R$, put

$$T_I = \mathcal{C}(R/I) \otimes \Bbbk(G)$$

Fract $\Bbbk[G]$

This is a commutative domain, a tensor product of two fields.

The rings T_I

For a given $I \in G\text{-}\mathrm{Spec}\, R$, put

$$T_I = \mathcal{C}(R/I) \otimes \Bbbk(G)$$

$\operatorname{Fract} \Bbbk[G]$

This is a commutative domain, a tensor product of two fields.

G-actions:

- on $\mathcal{C}(R/I)$ via the given action $\rho\colon G \to \mathrm{Aut}_{\Bbbk\text{-alg}}(R)$
- on $\Bbbk(G)$ by the right and left regular actions $\rho_r\colon (x.f)(y) = f(yx)$ and $\rho_\ell\colon (x.f)(y) = f(x^{-1}y)$
- on T_I by $\rho \otimes \rho_r$ and $\mathrm{Id} \otimes \rho_\ell$ $\longleftarrow$ commute

The rings T_I

For a given $I \in G\text{-}\operatorname{Spec} R$, put

Fract $\Bbbk[G]$

$$T_I = \mathcal{C}(R/I) \otimes \Bbbk(G)$$

This is a commutative domain, a tensor product of two fields.

Put

$$\operatorname{Spec}^G T_I = \{(\rho \otimes \rho_r)(G)\text{-stable primes of } T_I\}$$

Stratification Theorem

Theorem 3 *There is a bijection*

$$c\colon \operatorname{Spec}_I R \longrightarrow \operatorname{Spec}^G T_I$$

having the following properties:

(a) G-equivariance*:* $c(g.P) = (\mathrm{Id} \otimes \rho_\ell)(g)(c(P))$*;*

(b) inclusions*:* $P \subseteq P' \iff c(P) \subseteq c(P')$*;*

(c) hearts*:* $\mathcal{C}\,(T_I/c(P)) \cong \mathcal{C}\,(R/P \otimes \Bbbk(G))$ *as* $\Bbbk(G)$*-fields;*

(d) rationality*:* P *is rational* $\iff T_I/c(P) = \Bbbk(G)$*.*

Stratification Theorem

Theorem 3 *There is a bijection*

$$c\colon \operatorname{Spec}_I R \longrightarrow \operatorname{Spec}^G T_I$$

having the following properties:

(a) G-equivariance*:* $c(g.P) = (\mathrm{Id} \otimes \rho_\ell)(g)(c(P))$*;*

(b) inclusions*:* $P \subseteq P' \iff c(P) \subseteq c(P')$*;*

(c) hearts*:* $\mathcal{C}\,(T_I/c(P)) \cong \mathcal{C}\,(R/P \otimes \Bbbk(G))$ *as* $\Bbbk(G)$*-fields;*

(d) rationality*:* P *is rational* $\iff T_I/c(P) = \Bbbk(G)$*.*

Cor: Rational ideals are maximal in their strata

Rudolf Rentschler (PhD 1967 Munich)