Koszul algebras
and
MacMahon's Master Theorem

"Noncommutative Algebraic Geometry", Shanghai 09/20/2006

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Overview

- **MacMahon's "Master Theorem"**: statement, some background, references, . . .

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- Application: a new proof of the **quantum Master Theorem**

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- **Koszul algebras**: a quick introduction
- Application: a new proof of the **quantum Master Theorem**
- Recent developments

Objective

I will explain a **new algebraic approach**, obtained jointly with **Phùng Hô Hai** (Univ. of Duisburg-Essen and Inst. of Math., Hanoi), to the

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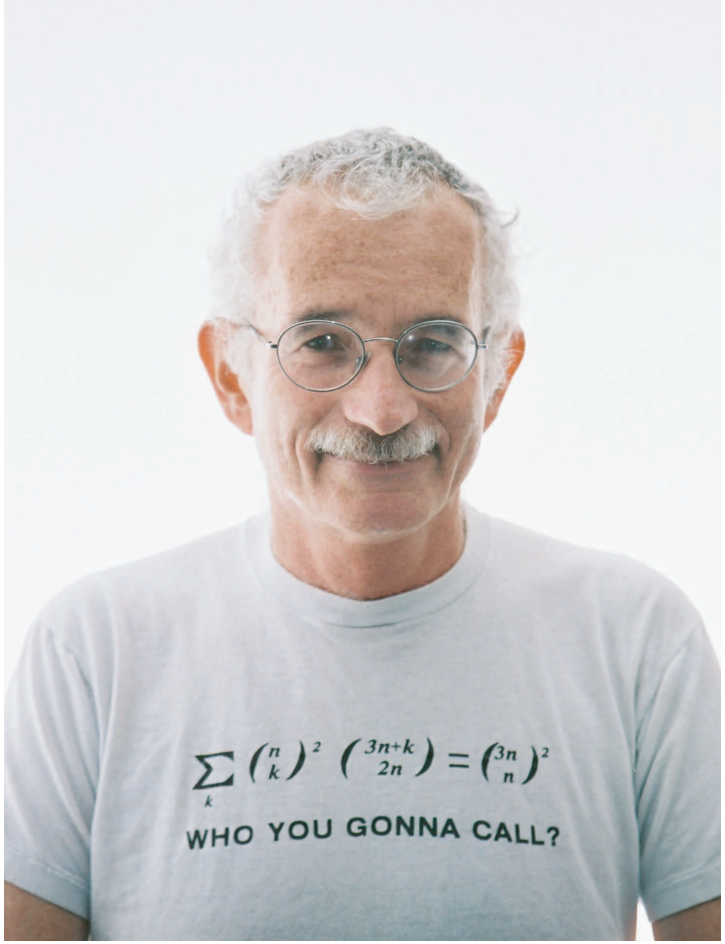


Doron Zeilberger

Rutgers University

Objective

The precise formulation of qMMT will be given later

– according to [GLZ], qMMT is *"a key ingredient in a finite non-commutative formula for the colored Jones polynomial of a knot"* –

but here is the original MMT . . .

Objective

MacMahon's Master Theorem (original version, 1917):

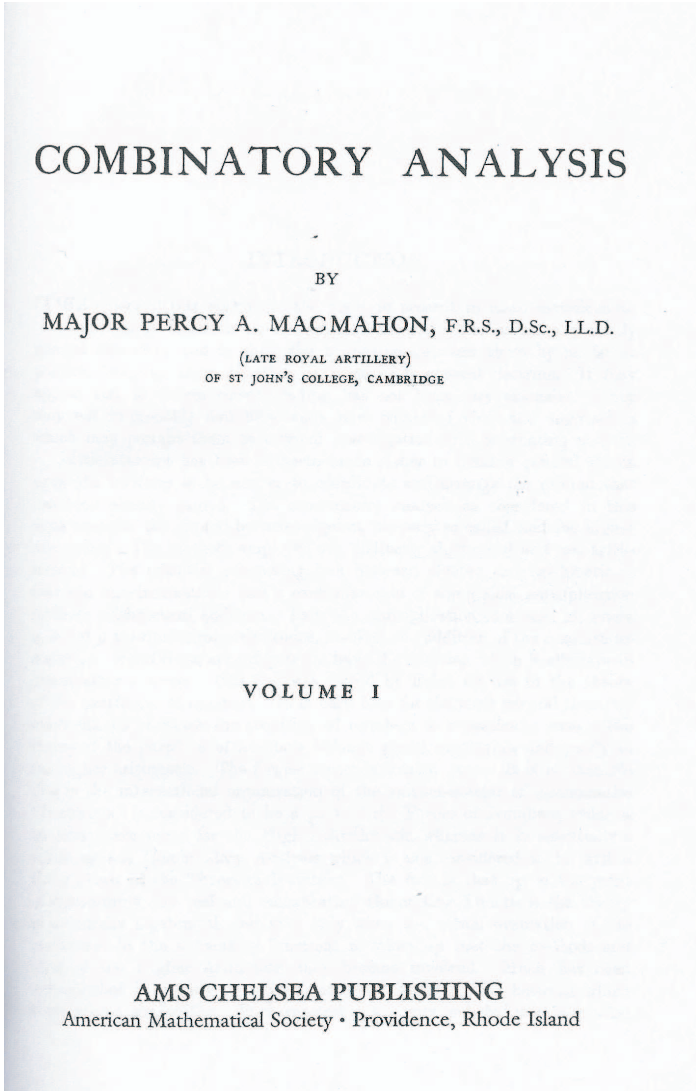
Given a matrix $A = (a_{ij})_{n\times n}$ *over some commutative ring* R *and commuting indeterminates* $x_1, \ldots, x_n$ *over* R*. For each* $(m_1, \ldots, m_n) \in \mathbb{Z}^n_{\geq 0}$*, the* R*-coefficient of* $x_1^{m_1} x_2^{m_2} \ldots x_n^{m_n}$ *in*

$$\left(\sum_{j=1}^{n} a_{1j}x_j\right)^{m_1} \left(\sum_{j=1}^{n} a_{2j}x_j\right)^{m_2} \cdots \left(\sum_{j=1}^{n} a_{nj}x_j\right)^{m_n}$$

is identical to the corresponding coefficient in

$$\det\left(1_{n\times n} - A\begin{pmatrix} x_1 & & \\ & \ddots & \\ & & x_n \end{pmatrix}\right)^{-1}$$

Background


COMBINATORY ANALYSIS

BY

MAJOR PERCY A. MACMAHON, F.R.S., D.Sc., LL.D.

(LATE ROYAL ARTILLERY)
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VOLUME I

AMS CHELSEA PUBLISHING
American Mathematical Society • Providence, Rhode Island

Percy Alexander MacMahon

1854 - 1929

Title page of MacMahon's book containing the "Master Theorem"

(originally published at Cambridge, 1917)

Background

... and here is where the name comes from:

wherein the denominator is in symbolic form in such wise that on multiplication the factors a_1b_2, $a_1b_2c_3$, ... are to be placed in determinant brackets $|a_1b_2|$, $|a_1b_2c_3|$, ... and denote the co-axial minors of the determinant

$$|a_1b_2 \dots n_n|,$$

which appertains to the matricular relation."

This is a master theorem in the Theory of Permutations.

We will write

$$V_n$$

for the expression

$$|(1-a_1x_1)(1-b_2x_2)\dots(1-n_nx_n)|,$$

and the reader will be able to verify that V_n has also the expression

$$(-)^n x_1x_2\dots x_n \begin{vmatrix} a_1 - 1/x_1 & a_2 & \cdot & \cdot & \cdot & a_n \\ b_1 & b_2 - 1/x_2 & \cdot & \cdot & \cdot & b_n \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ n_1 & n_2 & \cdot & \cdot & \cdot & n_n - 1/x_n \end{vmatrix}$$

Applications of the Theorem.

Background

Some later proofs:

- I. J. Good, *A short proof of MacMahon's 'Master Theorem'*, Math. Proc. Cambridge Philos. Soc. **58** (1960), 160

  $\rightsquigarrow$ **analysis**: contour integration
- Chu Wenchang, *Determinant, permanent, and MacMahon's Master Theorem*, Lin. Alg. and Appl. **255** (1997), 171–183

  $\rightsquigarrow$ **combinatorics**: cycle-generating functions, binomial identities
- I-C. Huang, *Applications of residues to combinatorial identies*, Proc. Amer. Math. Soc. **125** (1997), 1011–1017

  $\rightsquigarrow$ **algebra**: Grothendieck duality

Andrews' Problem:

(from George E. Andrews, *Problems and prospects for basic hypergeometric functions*, In: Theory and application of special functions, Academic Press, New York, 1975, pp. 191–224.)

5. MacMahon's Master Theorem and the Dyson Conjecture.

PROBLEM 5. Are there q-analogs of MacMahon's Master Theorem and the Dyson Conjecture?

First let us recall:

MacMahon's Master Theorem (MacMahon (1894), (1915)). The coefficient of $X_1^{P_1} X_2^{P_2} \ldots X_n^{P_n}$ in

Background

GLZ proved their qMMT in response to this problem.

The GLZ-quantization is **not** the first non-commutative version of MMT – Foata proved one as early as 1965 –

GLZ claim their quantization to be **natural**

Background

This talk is to support this claim

Reformulation of MMT

Recall: $A = (a_{ij}) \in \mathrm{Mat}_{n\times n}(R)$

$x_1, \dots, x_n$ commuting indeterminates over R

For each $\mathbf{m} = (m_1, \dots, m_n) \in \mathbb{Z}^n_{\geq 0}$, let the R-coefficient of $\mathbf{x}^{\mathbf{m}} = x_1^{m_1} x_2^{m_2} \dots x_n^{m_n}$ in $\prod_{i=1}^n \left(\sum_{j=1}^n a_{ij}x_j\right)^{m_i} \in R[x_1, \dots, x_n]$ be denoted by $c_A(\mathbf{m})$

MMT: $$1 = \det\left(1_{n\times n} - A\begin{pmatrix} x_1 & & \\ & \ddots & \\ & & x_n \end{pmatrix}\right) \cdot \sum_{\mathbf{m}} c_A(\mathbf{m})\mathbf{x}^{\mathbf{m}}$$

This is an identity in $R[\![x_1, \dots, x_n]\!]$

Reformulation of MMT

MMT: $$1 = \det\left(1_{n\times n} - A\begin{pmatrix} x_1 & & \\ & \ddots & \\ & & x_n \end{pmatrix}\right) \cdot \sum_{\mathbf{m}} c_A(\mathbf{m})\mathbf{x}^{\mathbf{m}}$$

all $x_i \mapsto t$ $\Downarrow$ $\Uparrow$ choose A "generic"

MMT': $$1 = \det\left(1_{n\times n} - At\right) \cdot \sum_{d=0}^{\infty}\left(\sum_{|\mathbf{m}|=d} c_A(\mathbf{m})\right) t^d$$

$= \sum_{d=0}^{n} \mathrm{trace}(\bigwedge^d A)(-t)^d$

$= \mathrm{trace}(\mathsf{S}^d A)$

Reformulation of MMT

To summarize, we have the following modern interpretation of MMT:

MMT

$$\Updownarrow$$

$$\boxed{1 = \left(\sum_{d=0}^{n} \mathrm{trace}(\bigwedge^d A)(-t)^d\right) \cdot \left(\sum_{d=0}^{\infty} \mathrm{trace}(\mathsf{S}^d A)t^d\right)}$$

All this is a well-known

Next: **Koszul algebras**

my main reference: **Yu. I. Manin**, *Quantum groups and noncommutative geometry*, Université de Montréal Centre de Recherches Mathématiques, Montreal, QC, 1988.

Quadratic algebras

Def: A **quadratic algebra** is a factor of the tensor algebra $\mathsf{T}(V)$ of some finite-dimensional $\Bbbk$-vector space V modulo quadratic relations:

$$\mathcal{A} \cong \mathsf{T}(V)/(R), \quad R \subseteq \mathsf{T}(V)_2 = V^{\otimes 2}$$

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The natural grading of $\mathsf{T}(V)$ descends to a **grading** of A:

$$\mathcal{A} = \bigoplus_{d \geq 0} \mathcal{A}_d \text{ with } \mathcal{A}_0 = \Bbbk,\ \mathcal{A}_1 \cong V$$

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$$\mathcal{A} \cong \mathsf{T}(V)/(R)\,, \quad R \subseteq \mathsf{T}(V)_2 = V^{\otimes 2}$$

Notation:

- $\mathcal{A} = A(V,R)$
- $\tilde{x}_1, \ldots, \tilde{x}_n$ will be a $\Bbbk$-basis of V
  $\rightsquigarrow \mathsf{T}(V) = \Bbbk\langle \tilde{x}_1, \ldots, \tilde{x}_n \rangle$, the free algebra
  $x_i := \tilde{x}_i \mod R$, algebra generators for $\mathcal{A}$

Quadratic algebras

Example: Quantum affine n-space

For fixed scalars $0 \neq q_{ij} \in \Bbbk$ $(1 \leq i < j \leq n)$, define

$$A_{\mathbf{q}}^{n|0} := \Bbbk\langle \tilde{x}_1, \ldots, \tilde{x}_n \rangle / \left(\tilde{x}_j \tilde{x}_i - q_{ij} \tilde{x}_i \tilde{x}_j \mid 1 \leq i < j \leq n \right)$$

So $A_{\mathbf{q}}^{n|0}$ is generated by $x_1, \ldots, x_n$ subject to the relations

$$x_j x_i = q_{ij} x_i x_j \quad \text{for } i < j.$$

Quadratic dual

Def: The **quadratic dual** of $\mathcal{A} = A(V, R)$ is defined by

$$\boxed{\mathcal{A}^! = A(V^*, R^\perp)}$$

with $R^\perp = \{f \in \left(V^{\otimes 2}\right)^* \cong (V^*)^{\otimes 2} \mid f(R) = 0\}$

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Notation: $\tilde{x}_1, \ldots, \tilde{x}_n$ a $\Bbbk$-basis of V, as before
$\tilde{x}^1, \ldots, \tilde{x}^n$ is the dual basis of V^*
$\rightsquigarrow$ generators $x^i = \tilde{x}^i \mod R^\perp$ for $\mathcal{A}^!$

Quadratic dual

Example: Quantum exterior algebra

The dual of quantum space $A_{\mathbf{q}}^{n|0}$ is denoted by $A_{\mathbf{q}}^{0|n}$

The procedure described yields algebra generators $x^1, \ldots, x^n$ for $A_{\mathbf{q}}^{0|n}$ satisfying the defining relations

$$x^\ell x^\ell = 0 \quad \text{for all } \ell$$

and

$$x^k x^\ell + q_{k\ell} x^\ell x^k = 0 \quad \text{for } k < \ell$$

The category $\mathsf{QAlg}_{\mathbb{k}}$

objects:	quadratic algebras$/\mathbb{k}$
morphisms:	graded algebra maps

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objects: quadratic algebras$/\mathbb{k}$
morphisms: graded algebra maps

Some further **operations on** $\mathsf{QAlg}_{\mathbb{k}}$:

- ordinary tensor product $\mathcal{A} \otimes \mathcal{B}$
- Segre product $\mathcal{A} \circ \mathcal{B} = \bigoplus_n \mathcal{A}_n \otimes \mathcal{B}_n$
- for $\mathcal{A} = A(V, R)$ and $\mathcal{B} = A(W, S)$, one has

$$\mathcal{A} \bullet \mathcal{B} = A(V \otimes W, \sigma_{23}(R \otimes S)\}$$

with $\sigma_{23} \colon V^{\otimes 2} \otimes W^{\otimes 2} \to (V \otimes W)^{\otimes 2}$ the $(2,3)$-switch

The category $\mathsf{QAlg}_{\mathbb{k}}$

objects: quadratic algebras $/\mathbb{k}$

morphisms: graded algebra maps

$\mathsf{QAlg}_{\mathbb{k}}^{\mathsf{op}}$ is the category of "**quantum linear spaces**" $/\mathbb{k}$

Analogies:

$\bullet \overset{!}{\longleftrightarrow} \circ$ tensor product of quantum spaces

$\otimes$ direct sum of quantum spaces

$!$ dualization plus parity change

The bialgebra $\underline{\text{end}}\,\mathcal{A}$

Def: For a given quadratic algebra $\mathcal{A} = A(V, R)$, Manin defines

$$\boxed{\underline{\text{end}}\,\mathcal{A} = \mathcal{A}^! \bullet \mathcal{A}}$$

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So $\underline{\mathrm{end}}\,\mathcal{A} = A(V^* \otimes V, \sigma_{23}(R^\perp \otimes R))$

Notation: $\tilde{x}_i, \tilde{x}^j$ dual bases for V and V^* as before

$\rightsquigarrow \tilde{z}_i^j = \tilde{x}^j \otimes \tilde{x}_i$ a basis of $V^* \otimes V$

$\rightsquigarrow z_i^j = \tilde{z}_i^j \mod R(\underline{\mathrm{end}}\,\mathcal{A})$ generate $\underline{\mathrm{end}}\,\mathcal{A}$

The bialgebra $\underline{\mathrm{end}}\,\mathcal{A}$

Properties:

- $\underline{\mathrm{end}}\,\mathcal{A}$ is a **bialgebra** over $\Bbbk$, with comultiplication

$$\Delta\colon \underline{\mathrm{end}}\,\mathcal{A} \to \underline{\mathrm{end}}\,\mathcal{A} \otimes \underline{\mathrm{end}}\,\mathcal{A}\ , \qquad \Delta(z_i^j) = \sum_\ell z_i^\ell \otimes z_\ell^j$$

  and counit

$$\epsilon\colon \underline{\mathrm{end}}\,\mathcal{A} \to \Bbbk\ , \qquad \epsilon(z_i^j) = \delta_{i,j}$$

- $\mathcal{A}$ is a left $\underline{\mathrm{end}}\,\mathcal{A}$-**comodule algebra**; the coaction is

$$\delta_{\mathcal{A}}\colon \mathcal{A} \to \underline{\mathrm{end}}\,\mathcal{A} \otimes \mathcal{A}\ , \qquad \delta_{\mathcal{A}}(x_i) = \sum_j z_i^j \otimes x_j\ ,$$

The bialgebra $\underline{\mathrm{end}}\,\mathcal{A}$

Example: Right quantum matrices

This is the algebra $\underline{\mathrm{end}}\,A_{\mathbf{q}}^{n|0}$. Defining relations:

column relations: $z_j^\ell z_i^\ell = q_{ij}\, z_i^\ell z_j^\ell \quad$ (all $\ell, i < j$)

cross relations: $q_{ij} z_i^k z_j^\ell - q_{k\ell} z_j^\ell z_i^k = z_j^k z_i^\ell - q_{ij} q_{k\ell}\, z_i^\ell z_j^k$
$(i < j, k < \ell)$

No "row relations"; even $\underline{\mathrm{end}}\,A_{q_{ij}=1}^{n|0}$ is non-commutative!

The bialgebra $\underline{\mathrm{end}}\,\mathcal{A}$

The relations for the generators z_i^j of $\underline{\mathrm{end}}\,A_{\mathbf{q}}^{n|0}$ are exactly those used by GLZ to define "right quantum matrices"

- GLZ only consider the case $q_{ij} = q$
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$\rightsquigarrow$ "**generic right quantum matrix**" $Z = (z_i^j)_{n \times n}$

Any algebra map $\varphi \colon \underline{\mathrm{end}}\, A_{\mathbf{q}}^{n|0} \to R$ ("R-point" of the space defined by $\underline{\mathrm{end}}\, A_{\mathbf{q}}^{n|0}$) yields a right quantum matrix φZ over R

Koszul complexes

For any quadratic algebra $\mathcal{A}$, one has Koszul complexes

$$\mathsf{K}^{\ell,\bullet}(\mathcal{A}): \; 0 \to \mathcal{A}^{!\,*}_{\ell} \to \mathcal{A}^{!\,*}_{\ell-1} \otimes \mathcal{A}_1 \to \cdots \to \mathcal{A}^{!\,*}_{1} \otimes \mathcal{A}_{\ell-1} \to \mathcal{A}_{\ell} \to 0$$

for all $\ell \geq 0$; for details see [Manin].

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Example: For the symmetric algebra $\mathcal{A} = \mathsf{S}(V) = A^{n|0}_{q_{ij}=1}$, these are the familiar Koszul complexes

$$\ldots \longrightarrow \wedge^{\ell-i+1}(V) \otimes \mathsf{S}^{i-1}(V) \longrightarrow \wedge^{\ell-i}(V) \otimes \mathsf{S}^{i}(V) \longrightarrow \ldots$$

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for all $\ell \geq 0$; for details see [Manin].

Lemma 1 (PHH & L) *All* $\mathsf{K}^{\ell,\bullet}(\mathcal{A})$ *are complexes of* $\underline{\mathrm{end}}\,\mathcal{A}$*-comodules.*

Koszul complexes

For any quadratic algebra $\mathcal{A}$, one has Koszul complexes

$$\mathsf{K}^{\ell,\bullet}(\mathcal{A}): \; 0 \to \mathcal{A}^{!}_{\ell}{}^{*} \to \mathcal{A}^{!}_{\ell-1}{}^{*} \otimes \mathcal{A}_1 \to \cdots \to \mathcal{A}^{!}_{1}{}^{*} \otimes \mathcal{A}_{\ell-1} \to \mathcal{A}_\ell \to 0$$

for all $\ell \geq 0$; for details see [Manin].

Def: The quadratic algebra $\mathcal{A}$ is said to be **Koszul** iff the complexes $\mathsf{K}^{\ell,\bullet}(\mathcal{A})$ are exact for $\ell > 0$.

Some Koszul facts

- Koszul algebras were introduced by Stewart Priddy in connection with his investigation of Yoneda algebras $\mathrm{Ext}_A(\Bbbk, \Bbbk)$ (Trans AMS, 1970)

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- The class of Koszul algebras is quite robust: it is stable under the operations $!$, $\otimes$, $\circ$, $\bullet$, $\underline{\mathrm{end}}$, ...

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- The class of Koszul algebras is quite robust: it is stable under the operations $!$, $\otimes$, $\circ$, $\bullet$, $\underline{\mathrm{end}}$, ...

- A sufficient condition for $\mathcal{A}$ to be Koszul is the existence of a **PBW-basis**.

Some Koszul facts

Example: $A_{\mathbf{q}}^{n|0}$ and right quantum matrices

Recall that $A_{\mathbf{q}}^{n|0}$ is generated $x_1, \ldots, x_n$ subject to the relations

$$x_j x_i = q_{ij} x_i x_j \quad \text{for } i < j.$$

The algebra $A_{\mathbf{q}}^{n|0}$ has a $\mathbb{k}$-basis consisting of the ordered monomials $\mathbf{x}^{\mathbf{m}} = x_1^{m_1} x_2^{m_2} \ldots x_n^{m_n}$; this is a PBW-basis.

$\Longrightarrow$ $A_{\mathbf{q}}^{n|0}$ is Koszul (and also $A_{\mathbf{q}}^{0|n}$, $\underline{\mathrm{end}}\, A_{\mathbf{q}}^{n|0}$ ...)

Characters

Notation: $\mathcal{B}$ some bialgebra over $\mathbb{k}$ (later: $\mathcal{B} = \underline{\text{end}}\, \mathcal{A}$)

$R_{\mathcal{B}}$ Grothendieck ring of all left B-comodules that are finite-dimensional$/\mathbb{k}$ (or f.g. projective)

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In more detail:

- $\mathcal{B}$-comodule $V \quad \rightsquigarrow \quad [V] \in R_{\mathcal{B}}$
- $0 \to U \to V \to W \to 0$ exact $\quad \rightsquigarrow \quad [V] = [U] + [W]$ in $R_{\mathcal{B}}$
- Multiplication in $R_{\mathcal{B}}$ is given by the tensor product of B-comodules

Characters

Def: Let V be a $\mathcal{B}$-comodule; so have $\delta_V : V \to \mathcal{B} \otimes V$
Consider the map

$$\mathrm{Hom}_{\Bbbk}(V, \mathcal{B} \otimes V) \cong \mathcal{B} \otimes V \otimes V^* \xrightarrow{\mathrm{Id}_{\mathcal{B}} \otimes \langle\, .\, ,\, .\, \rangle} \mathcal{B} \otimes \Bbbk \cong \mathcal{B}$$

evaluation $V \otimes V^* \to \Bbbk$

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Explicitly: If $\delta_V(v_j) = \sum_i b_{i,j} \otimes v_i$ for some $\Bbbk$-basis $\{v_i\}$ of V then

$$\chi_V = \sum_i b_{i,i}$$

Characters

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Lemma 2 *The map $[V] \mapsto \chi_V$ yields a well-defined ring homomorphism $\chi\colon R_{\mathcal{B}} \to \mathcal{B}$.*

MMT for Koszul algebras

Recall the modern interpretation of the original MMT:

$$1 = \left(\sum_{d=0}^{n} \operatorname{trace}(\bigwedge^{d} A)(-t)^{d} \right) \cdot \left(\sum_{d=0}^{\infty} \operatorname{trace}(\mathsf{S}^{d} A) t^{d} \right)$$

for any $n \times n$-matrix A over some commutative ring

MMT for Koszul algebras

Here is the version for Koszul algebras:

Theorem 1 ($\text{PHH \& L}$) *Let $\mathcal{A}$ be a Koszul algebra and $\mathcal{B} = \underline{\mathrm{end}}\,\mathcal{A}$. Then the following identity holds in $\mathcal{B}[\![t]\!]$:*

$$1 = \left(\sum_{m \geq 0} \chi_{\mathcal{A}^{!\,*}_m} (-t)^m \right) \cdot \left(\sum_{\ell \geq 0} \chi_{\mathcal{A}_\ell} t^\ell \right)$$

MMT for Koszul algebras

Proof: By Lemma 1, the (exact) Koszul complexes

$$\mathsf{K}^{\ell,\bullet}(\mathcal{A}):\; 0 \to \mathcal{A}^{!\,*}_{\ell} \to \mathcal{A}^{!\,*}_{\ell-1} \otimes \mathcal{A}_1 \to \cdots \to \mathcal{A}^{!\,*}_{1} \otimes \mathcal{A}_{\ell-1} \to \mathcal{A}_{\ell} \to 0$$

give equations in $R_{\mathcal{B}}$:

$$\sum_i (-1)^i [\mathcal{A}^{!\,*}_i][\mathcal{A}_{\ell-i}] = 0 \qquad (\ell > 0)$$

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give equations in $R_{\mathcal{B}}$:

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Defining $P_{\mathcal{A}}(t) = \sum_i [\mathcal{A}_i] t^i$, $P_{\mathcal{A}^{!*}}(t) = \sum_i [\mathcal{A}^{!\,*}_i] t^i \in R_{\mathcal{B}}[\![t]\!]$, this becomes

$$1 = P_{\mathcal{A}^{!*}}(-t) \cdot P_{\mathcal{A}}(t)$$

MMT for Koszul algebras

Proof: By Lemma 1, the (exact) Koszul complexes

$$\mathsf{K}^{\ell,\bullet}(\mathcal{A}):\ 0 \to \mathcal{A}^{!}_{\ell}{}^{*} \to \mathcal{A}^{!}_{\ell-1}{}^{*} \otimes \mathcal{A}_1 \to \cdots \to \mathcal{A}^{!}_{1}{}^{*} \otimes \mathcal{A}_{\ell-1} \to \mathcal{A}_\ell \to 0$$

give equations in $R_{\mathcal{B}}$:

$$\sum_i (-1)^i [\mathcal{A}^{!}_{i}{}^{*}][\mathcal{A}_{\ell-i}] = 0 \qquad (\ell > 0)$$

Defining $P_{\mathcal{A}}(t) = \sum_i [\mathcal{A}_i] t^i, P_{\mathcal{A}^{!*}}(t) = \sum_i [\mathcal{A}^{!}_{i}{}^{*}] t^i \in R_{\mathcal{B}}[\![t]\!]$, this becomes

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Now apply the ring homomorphism $\chi[\![t]\!]: R_{\mathcal{B}}[\![t]\!] \to \mathcal{B}[\![t]\!]$. QED

qMMT

Garoufalidis, Lê and Zeilberger's qMMT is **exactly** the special case of Theorem 1 where $\mathcal{A} = A_q^{n|0}$

Spelled out in detail . . .

(multiparameter version)

qMMT

Notation:

(as before)

$$\mathcal{A} = A_{\mathbf{q}}^{n|0} = \Bbbk[x_i \mid i = 1, \ldots, n]$$
$$\mathcal{B} = \underline{\text{end}}\, A_{\mathbf{q}}^{n|0} = \Bbbk[z_i^j \mid i, j = 1, \ldots, n]$$
$$Z = (z_i^j)_{n \times n}$$

Further,

$$\det_{\mathbf{q}}(Z) = \sum_{\pi \in \mathfrak{S}_n} w(\pi) z_{\pi 1}^1 z_{\pi 2}^2 \ldots z_{\pi n}^n$$

is the multiparameter **quantum determinant** as defined by [AST], with $w(\pi) = \prod_{i<j,\, \pi i > \pi j} \left(-q_{\pi j, \pi i}\right)^{-1}$

Finally, for each $J \subseteq \{1, \ldots, n\}$, I will write $Z_J = (z_i^j)_{i,j \in J}$.

qMMT

Theorem 2 (qMMT) In $\mathcal{B} \otimes \mathcal{A} = \bigoplus_{\mathbf{m} \in \mathbb{Z}^n_{\geq 0}} \mathcal{B} \otimes \mathbf{x}^{\mathbf{m}}$ put $X_i = \sum_j z_i^j \otimes x_j$ and define $G(\mathbf{m})$ to be the $\mathcal{B}$-coefficient of $\mathbf{x}^{\mathbf{m}}$ in $X_1^{m_1} X_2^{m_2} \dots X_n^{m_n}$. In $\mathcal{B}[\![t]\!]$ put

$$\text{Bos}(Z) := \sum_{\ell \geq 0} \sum_{|\mathbf{m}| = \ell} G(\mathbf{m}) t^\ell$$

$$\text{Ferm}(Z) := \sum_{m \geq 0} \sum_{\substack{J \subseteq \{1, \dots, n\} \\ |J| = m}} \det_{\mathbf{q}}(Z_J)(-t)^m$$

Then:

$$\boxed{\text{Bos}(Z) \cdot \text{Ferm}(Z) = 1}$$

qMMT

In view of Theorem 1, the **proof** of Theorem 2 amounts to two character calculations:

- $$\chi_{\mathcal{A}_\ell} = \sum_{|\mathbf{m}|=\ell} G(\mathbf{m})$$
- $$\chi_{\mathcal{A}^{!*}_m} = \sum_{\substack{J \subseteq \{1,\dots,n\} \\ |J|=m}} \det_{\mathbf{q}}(Z_J)$$

Both are easy.

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- Zeilberger has also written `Maple` programs `QuantumMACMAHON` and `qMM` that verify **qMMT**
  (available on Zeilberger's web page)
- Independent work by **Foata & Han** gives an alternative new proof of **qMMT** using combinatorics on words. They also analyze the algebra of right quantum matrices in detail and give various modifications of **qMMT**
  (3 preprints, December 2005, available on arXiv)

Recent developments

- Konvalinka and Pak use a **combinatorial approach** to derive the above multiparameter qMMT (for the first time) as well as various other (super-)versions of MMT.

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- Jointly with Benoit Kriegk (Saint-Étienne) and Phùng Hô Hai, I am currently writing up an extension of the foregoing to "N-homogeneous Koszul superalgebras".