Multiplicative Invariant Theory

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1^{st} ***talk***

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Overview

- **Multiplicative invariants**: definitions, examples, historical roots, . . .

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- **Cohen-Macaulay rings**: a quick introduction and some recent results on multiplicative invariants

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- **Multiplicative invariants**: definitions, examples, historical roots, . . .
- **Cohen-Macaulay rings**: a quick introduction and some recent results on multiplicative invariants
- Some **open problems**

Multiplicative Invariants

- Given: a group G and a G-**lattice** $L \cong \mathbb{Z}^n$; so

$$G \longrightarrow \mathrm{GL}(L) \cong \mathrm{GL}_n(\mathbb{Z})$$

an integral representation of G

Multiplicative Invariants

- Given: a group G and a G-**lattice** $L \cong \mathbb{Z}^n$; so

$$G \longrightarrow \mathrm{GL}(L) \cong \mathrm{GL}_n(\mathbb{Z})$$

- Choose a base ring $\Bbbk$ and form the **group algebra**

$$\Bbbk[L] = \bigoplus_{m \in L} \Bbbk \mathbf{x}^m \cong \Bbbk[x_1^{\pm 1}, \dots, x_n^{\pm 1}] , \qquad \mathbf{x}^m \mathbf{x}^{m'} = \mathbf{x}^{m+m'}$$

The G-action on L extends uniquely to a "multiplicative" action by $\Bbbk$-algebra automorphisms on $\Bbbk[L]$:

$$\boxed{g(\mathbf{x}^m) = \mathbf{x}^{g(m)} \qquad (g \in G, m \in L)}$$

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  The G-action on L extends uniquely to a "multiplicative" action by $\Bbbk$-algebra automorphisms on $\Bbbk[L]$.

- The multiplicative invariant algebra is

$$\boxed{\Bbbk[L]^G = \{ f \in \Bbbk[L] \mid g(f) = f \ \forall g \in G \}}$$

Example #1

Multiplicative invariants of the standard permutation lattice:

$\mathcal{S}_n$ is the symmetric group
$U_n = \bigoplus_1^n \mathbb{Z}e_i \cong \mathbb{Z}^n$
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Put $x_i = \mathbf{x}^{e_i} \in \Bbbk[U_n]$; so $\sigma(x_i) = x_{\sigma(i)}$ for $\sigma \in \mathcal{S}_n$. Then

$$\Bbbk[U_n] = \Bbbk[x_1^{\pm 1}, \ldots, x_n^{\pm 1}] = \Bbbk[x_1, \ldots, x_n][s_n^{-1}] \,,$$

where $s_n = \prod_1^n x_i$ is the n^{th} elementary symmetric polynomial.

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action: $\sigma(e_i) = e_{\sigma(i)}$ $(\sigma \in \mathcal{S}_n)$

$$\therefore \qquad \begin{aligned} \Bbbk[U_n]^{\mathcal{S}_n} &= \Bbbk[x_1, \ldots, x_n][s_n^{-1}]^{\mathcal{S}_n} \\ &= \Bbbk[x_1, \ldots, x_n]^{\mathcal{S}_n}[s_n^{-1}] \\ &= \Bbbk[s_1, \ldots, s_{n-1}, s_n^{\pm 1}] \\ &\cong \Bbbk[\mathbb{Z}_+^{n-1} \oplus \mathbb{Z}] \end{aligned}$$

elem. symmetric poly's

Special Features

Back to general multiplicative actions:

L	a G-lattice
$\Bbbk$	a commutative base ring
$\Bbbk[L]$	the group algebra

Special Features

$\mathbb{Z}$**-structure**:

$$\mathbb{k}[L]^G = \mathbb{k} \otimes_{\mathbb{Z}} \mathbb{Z}[L]^G$$

Special Features

It suffices to consider **finite** groups:

Special Features

Put

$$L_{\text{fin}} = \{m \in L \mid [G : G_m] < \infty\} \; .$$

stabilizer of $m \in L$

G acts on L_{fin} through the finite quotient $\mathcal{G} = G / \operatorname{Ker}_G(L_{\text{fin}})$, and

$$\boxed{\Bbbk[L]^G = \Bbbk[L_{\text{fin}}]^{\mathcal{G}}}$$

Special Features

In particular, $\mathbb{k}[L]^G$ is **always affine**/$\mathbb{k}$ (Hilbert # 14 ok).

Special Features

In general, $\mathbb{k}[L]$ has **no grading** (connected) that is preserved by the action of G.

$\rightsquigarrow$ computational theory not yet highly developed

$\exists$ some GAP & MAGMA-programs (L., Marc Renault)

Example #2

The "weight lattice" $A_{n-1}^* = U_n/\mathbb{Z}(e_1 + \cdots + e_n) \cong \mathbb{Z}^{n-1}$

($\mathcal{S}_n$ and $U_n = \bigoplus_1^n \mathbb{Z}e_i$ as in Example #1)

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$\big(\mathcal{S}_n$ and $U_n = \bigoplus_1^n \mathbb{Z}e_i$ as in Example #1$\big)$

So

$$
\begin{aligned}
U_n \twoheadrightarrow A_{n-1}^* \quad &\rightsquigarrow \quad \Bbbk[U_n] \twoheadrightarrow \Bbbk[A_{n-1}^*] \\
&\rightsquigarrow \quad \Bbbk[U_n]^{\mathcal{S}_n} \twoheadrightarrow \Bbbk[A_{n-1}^*]^{\mathcal{S}_n}
\end{aligned}
$$

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So

$$\begin{aligned} U_n \twoheadrightarrow A_{n-1}^* \quad &\rightsquigarrow \quad \mathbb{k}[U_n] \twoheadrightarrow \mathbb{k}[A_{n-1}^*] \\ &\rightsquigarrow \quad \mathbb{k}[U_n]^{\mathcal{S}_n} \twoheadrightarrow \mathbb{k}[A_{n-1}^*]^{\mathcal{S}_n} \end{aligned}$$

Under the last map, $\mathbf{x}^{e_1+\cdots+e_n} = s_n \quad \mapsto \quad 1$.

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So

$$U_n \twoheadrightarrow A_{n-1}^* \quad \rightsquigarrow \quad \Bbbk[U_n] \twoheadrightarrow \Bbbk[A_{n-1}^*]$$

$$\rightsquigarrow \quad \Bbbk[U_n]^{\mathcal{S}_n} \twoheadrightarrow \Bbbk[A_{n-1}^*]^{\mathcal{S}_n}$$

Under the last map, $\mathbf{x}^{e_1+\cdots+e_n} = s_n \quad \mapsto \quad 1.$

$$\therefore \qquad \Bbbk[A_{n-1}^*]^{\mathcal{S}_n} \cong \Bbbk[U_n]^{\mathcal{S}_n}/(s_n - 1)$$

$$= \Bbbk[s_1, \ldots, s_{n-1}, s_n^{\pm 1}]/(s_n - 1)$$

$$\cong \Bbbk[s_1, \ldots, s_{n-1}] \cong \Bbbk[\mathbb{Z}_+^{n-1}]$$

Example #3

The root lattice $A_{n-1} = \{\sum_i z_i e_i \in U_n \mid \sum_i z_i = 0\} \cong \mathbb{Z}^{n-1}$ (notation as before)

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Here,

$$\Bbbk[A_{n-1}] = \Bbbk[U_n]_0 \ ,$$

the degree 0-component for the ($\mathcal{S}_n$-stable) “total degree” grading of $\Bbbk[U_n] = \Bbbk[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$.

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Get

$$\Bbbk[A_{n-1}]^{\mathcal{S}_n} = \Bbbk[U_n]_0^{\mathcal{S}_n} \cong \Bbbk[M]$$

with

$$M = \{(t_1, \ldots, t_{n-1}) \in \mathbb{Z}_+^{n-1} \mid \sum i t_i \in n\mathbb{Z}\} \; ,$$

a submonoid of $\mathbb{Z}_+^{n-1}$.

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The root lattice $A_{n-1} = \{\sum_i z_i e_i \in U_n \mid \sum_i z_i = 0\} \cong \mathbb{Z}^{n-1}$
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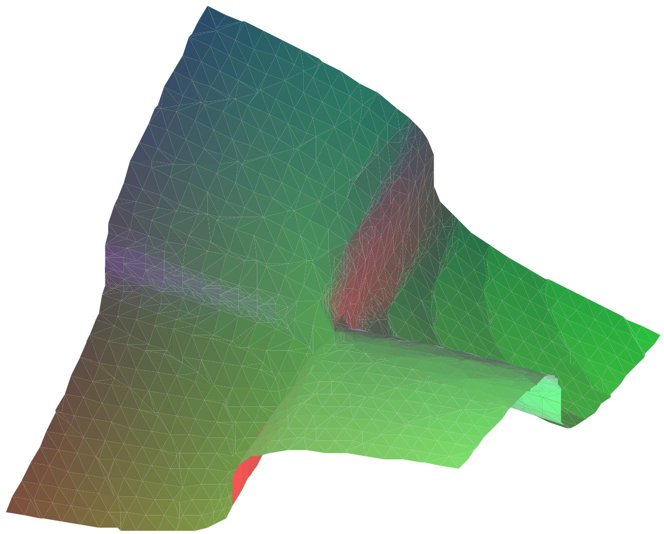
$\mathbb{C}[A_{n-1}]^{\mathcal{S}_n}$ is **not** regular:
($n > 2$; picture for $n = 3$)



Rational Equivalence

Two G-lattices L and L' are called **rationally isomorphic** if $L \otimes_{\mathbb{Z}} \mathbb{Q} \cong L' \otimes_{\mathbb{Z}} \mathbb{Q}$ as $\mathbb{Q}[G]$-modules.

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Rationally isomorphic lattices can have **very different** multiplicative invariant algebras . . .

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Example: The $\mathcal{S}_n$-lattices A_{n-1}^* and A_{n-1} are rationally isomorphic:

$$A_{n-1}^* \otimes_{\mathbb{Z}} \mathbb{Q} \cong A_{n-1} \otimes_{\mathbb{Z}} \mathbb{Q} \cong S^{(n-1,1)}$$

the Specht module for the partition $(n-1,1)$ of n. Yet,

- $\mathbb{Z}[A_{n-1}^*]^{\mathcal{S}_n}$ is a polynomial ring over $\mathbb{Z}$
- $\mathbb{Z}[A_{n-1}]^{\mathcal{S}_n}$ is not regular and not a UFD for $n > 2$.

Pioneers

- **Bourbaki**: *"Invariants exponentiels"*, in: *Groupes et algèbres de Lie* (1968).

$$\boxed{R(\mathfrak{g}) \cong \mathbb{Z}[\Lambda]^{\mathcal{W}} \cong \mathbb{Z}[x_1, \ldots, x_{\operatorname{rank}\mathfrak{g}}]}$$

where $R(\mathfrak{g})$ = representation ring of a semisimple Lie algebra $\mathfrak{g}$, Λ = weight lattice of $\mathfrak{g}$, and $\mathcal{W}$ = Weyl group.

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- **Steinberg, Richardson** (1970s)

- "Δ-methods" for group rings: **Passman**, **Zalesskiĭ**, **Roseblade**, **Dan Farkas** $\rightsquigarrow$ "multiplicative invariants" (mid 1980s)

Finite Linear Groups

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$\therefore$ there are only **finitely many** multiplicative invariant algebras $\Bbbk[L]^G$ (up to $\cong$) with $\operatorname{rank} L$ bounded

Finite Linear Groups

n	# fin. $\mathcal{G} \leq \mathrm{GL}_n(\mathbb{Z})$ (up to conj.)	# max'l $\mathcal{G}$ (up to conj.)
1	2	1
2	13	2
3	73	4
4	710	9
5	6079	17
6	85311	39

Finite Linear Groups

More on finite linear groups in the second talk . . .

Part II

Next: Cohen-Macaulay (CM) rings

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(Zariski) topology
dimension theory

(homological) algebra

- **Def:** R is **Cohen-Macaulay** iff equality holds for all (maximal) ideals $\mathfrak{a}$.

Some Examples of CM Rings

- **Standard example:** R an affine domain/PID $\Bbbk$, finite / some polynomial subalgebra $P = \Bbbk[x_1, \ldots, x_n]$. Then:

$$R \text{ CM} \Leftrightarrow R \text{ is free over } P$$

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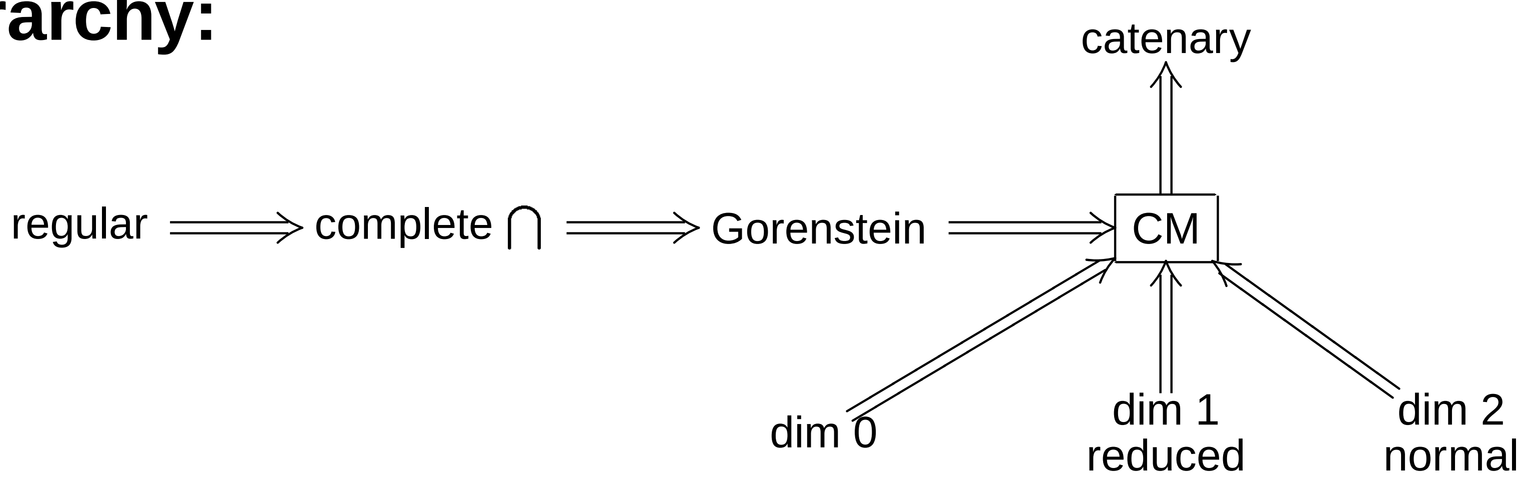
- **Hierarchy:**



Invariant Rings

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Here is a **necessary condition** . . .

Invariant Rings

Hypotheses:

R a CM ring

$\mathcal{G}$ a finite group acting on R

$\mathcal{R}_k = \{ k$-reflections on $R \}$

Assume R noetherian $/R^{\mathcal{G}}$

automorphisms belonging to the inertia group of some prime of height $\leq k$

Theorem 1 (L. - Pathak) *If $R^{\mathcal{G}}$ CM & $H^i(\mathcal{G}, R) = 0 \ (0 < i < k)$ then*

$$\mathrm{res}\colon H^k(\mathcal{G}, R) \hookrightarrow \prod_{\mathcal{H} \subseteq \mathcal{R}_{k+1}} H^k(\mathcal{H}, R)$$

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Note: The $(H^i = 0)$-cond^n is vacuous for $k = 1$ $\rightsquigarrow$ bireflections.

Invariant Rings

Main ingredients of proof:

- spectral sequences by **Ellingsrud & Skjelbred**:

$$\begin{array}{l} E_2^{p,q} = H^p_{\mathfrak{a}}(H^q(G,M)) \\ \qquad\qquad\qquad\qquad \Longrightarrow H^{p+q}_{\mathfrak{a}}(G,M) \\ \mathcal{E}_2^{p,q} = H^p(G, H^q_{\mathfrak{a}}(M)) \end{array}$$

- calculation of the closed set in $\operatorname{Spec} R^{\mathcal{G}}$ determined by the image of **transfer** $R^{\mathcal{H}} \to R^{\mathcal{G}}$ for $\mathcal{H} \leq \mathcal{G}$.

Multiplicative Invariants: CM-property

Notations: $\mathcal{G}$ is a finite group $\neq 1$
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So $\mathcal{G} \hookrightarrow \mathrm{GL}(L), g \mapsto g_L$. In this setting,

$g \in \mathcal{G}$ is a k-reflection on $\Bbbk[L]$	$\Longleftrightarrow$	$\mathrm{rank}(g_L - \mathrm{Id}_L) \leq k$

"g is a k-reflection on L" (or on $L \otimes_{\mathbb{Z}} \mathbb{Q}$)

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Decoding Theorem 1 and mixing it with some representation theory, notably a result of **Zassenhaus** on fixed-point-free complex representations of perfect groups, we obtain . . .

Multiplicative Invariants: CM-property

Notations: $\mathcal{G}$ is a finite group $\neq 1$
L a $\mathcal{G}$-lattice, WLOG faithful

Theorem 2 (L, Trans AMS '06) *Assume that $\mathbb{Z}[L]^{\mathcal{G}}$ is CM. Then all $\mathcal{G}_m/\mathcal{R}^2(\mathcal{G}_m)$ are perfect groups, but not all $\mathcal{G}_m$ are.*

stabilizer of $m \in L$ ($\mathcal{G}_m$)

subgroup gen. by bireflections on L ($\mathcal{R}^2(\mathcal{G}_m)$)

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Corollary ("3-copies conjecture") *$\mathbb{Z}[L^{\oplus r}]^{\mathcal{G}}$ is never CM for $r \geq 3$.*

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Note that the conclusions of Theorem 2 only refer to the **rational** type of L. In fact . . .

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Proposition *If $\Bbbk[L]^{\mathcal{G}}$ is CM then so is $\Bbbk[L']^{\mathcal{G}}$ for any $\mathcal{G}$-lattice L' rationally isomorphic to L.*

Example: $\mathcal{S}_n$-lattices

What are the $\mathcal{S}_n$-lattices L such that $\mathbb{Z}[L]^{\mathcal{S}_n}$ is CM ?

Example: $\mathcal{S}_n$-lattices

We know:

- only the structure of $L_{\mathbb{Q}} = L \otimes_{\mathbb{Z}} \mathbb{Q}$ matters (Proposition)
- $\mathcal{S}_n$ must act as a bireflection group on L (Theorem 2), and hence on all simple constituents of $L_{\mathbb{Q}}$

Example: $\mathcal{S}_n$-lattices

Classification results of irreducible finite linear groups containing a bireflection (Huffman and Wales, 70s) imply, for $n \geq 7$:

$$L_{\mathbb{Q}} \cong \mathbb{Q}^r \oplus \left(\mathbb{Q}^-\right)^s \oplus (A_{n-1})^t_{\mathbb{Q}} \qquad (s + t \leq 2)$$

sign representation of $\mathcal{S}_n$

Example: $\mathcal{S}_n$-lattices

In all cases, $\mathbb{Z}[L]^{\mathcal{S}_n}$ is indeed CM, with the possible exception of

$$L = A_{n-1}^2$$

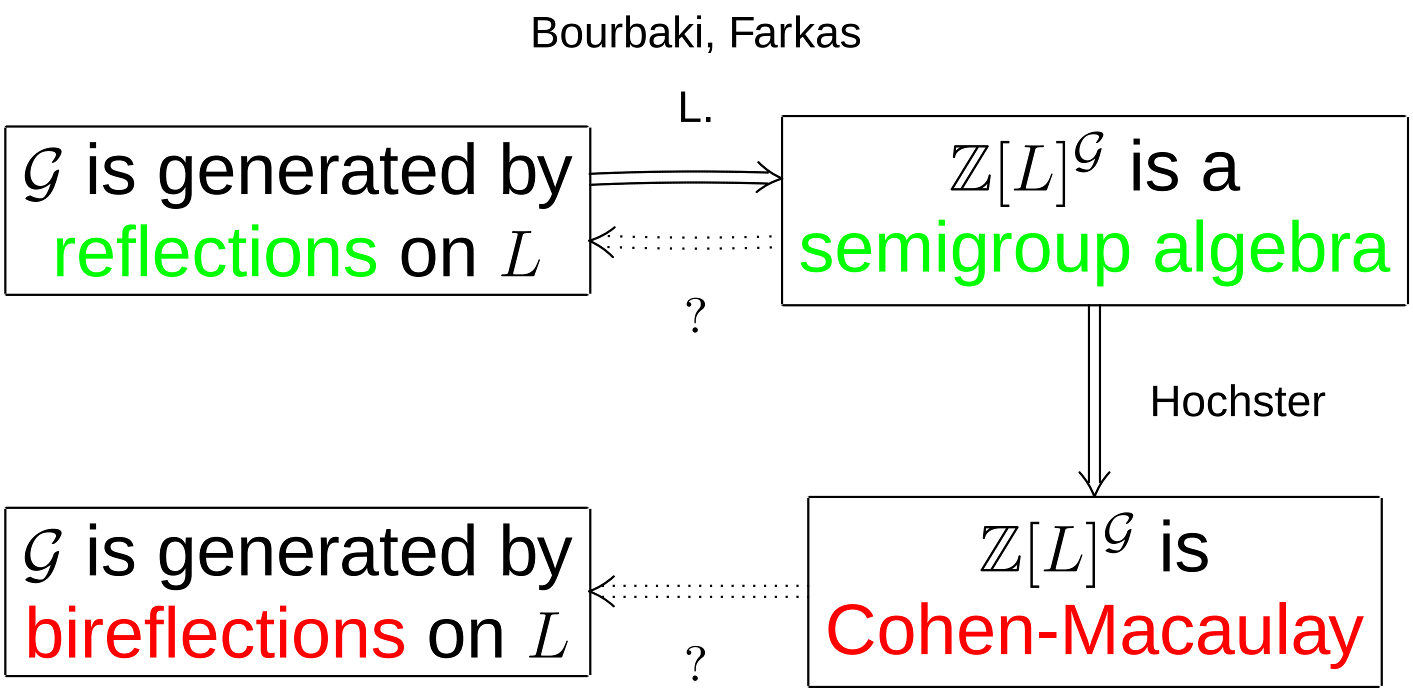
This case reduces to

Problem (open for $p \leq n/2$)	Are the "vector invariants" $\mathbb{F}_p[x_1, \ldots, x_n, y_1, \ldots, y_n]^{\mathcal{S}_n}$ CM?

Outlook

Multiplicative invariant theory

Let L be a $\mathcal{G}$-lattice, where $\mathcal{G}$ is a finite group.



Example #4

Multiplicative inversion in rank 2:
($\mathbb{k} = \mathbb{Z}$)

$\mathcal{C}_2 = \langle g \mid g^2 = 1 \rangle$
$L_2 = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2$
action: $g(e_i) = -e_i$

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So: $\mathbb{Z}[L_2] = \mathbb{Z}[x_1^{\pm 1}, x_2^{\pm 1}]$ with $g(x_i) = x_i^{-1}$

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action: $g(e_i) = -e_i$

Straightforward calculation gives

$$\mathbb{Z}[L_2]^{\mathcal{C}_2} = \mathbb{Z}[\xi_1, \xi_2] \oplus \eta\mathbb{Z}[\xi_1, \xi_2]$$

where $\xi_i = x_i + x_i^{-1}$ and $\eta = x_1 x_2 + x_1^{-1} x_2^{-1}$

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One observes $\eta\xi_1\xi_2 = \eta^2 + \xi_1^2 + \xi_2^2 - 4$. Hence,

$$\mathbb{Z}[L_2]^{\mathcal{C}_2} \cong \mathbb{Z}[x, y, z]/(x^2 + y^2 + z^2 - xyz - 4) \; .$$

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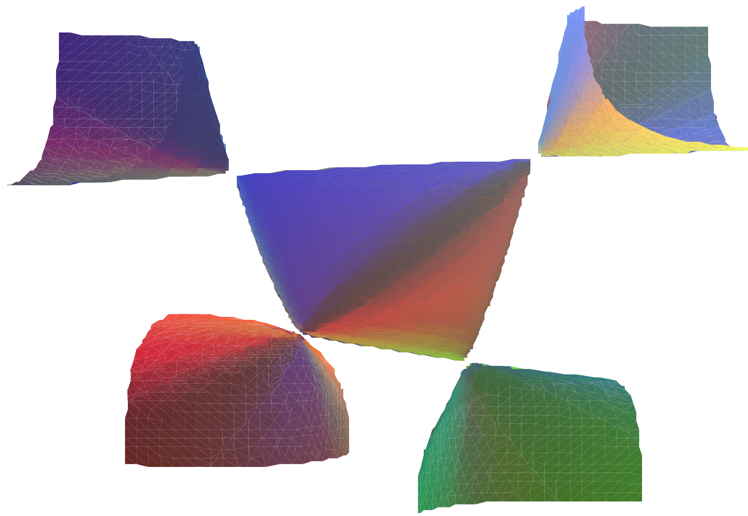
$\mathbb{Z}[L_2]^{\mathcal{C}_2}$ is **not** a semi-group algebra:



One more problem . . .

Linear groups

$\mathcal{G} \subseteq \mathrm{GL}(V)$ a finite linear group (V some vector space).

Question If $\mathcal{G}$ is generated by bireflections, is this also true for all $\mathcal{G}_v$ $(v \in V)$?

The answer is “NO" for general k-reflections (Zalesskiĭ), but it is “YES" for reflections ($k = 1$; Steinberg, Serre).

One more problem . . .

?