Some applications of Frobenius algebras to Hopf algebras

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Overview

(A) **Ring theory**:

- Background on Frobenius algebras

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- Separability locus

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(B) **Applications**:

- Representations of Hopf algebras

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- Background on Frobenius algebras
- Separability locus
- Characters and integrality

(B) **Applications**:

- Representations of Hopf algebras
- A problem . . . time permitting

Reference

- *"Some applications of Frobenius algebras to Hopf algebras"* preprint (Dec. 2009)

Article & **pdf file of this talk** available on my web page:

http://math.temple.edu/~lorenz/

Frobenius algebras

Notation

For the remainder of this talk,

R	denotes a commutative ring
A	is an associative R-algebra that is fin. gen. projective ("finite") over R

Definitions

Put $A^\vee = \mathrm{Hom}_R(A, R)$; this is an (A, A)-bimodule via the (A, A)-bimodule structure on A:

$$(afb)(x) = f(bxa) \qquad (a, b, x \in A, f \in A^\vee) .$$

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The algebra A is called Frobenius if it has the following equivalent properties

- $A \cong A^\vee$ as right A-modules;
- there exists a nonsingular R-bilinear form $\beta\colon A \times A \to R$ that is associative: $\beta(ab, c) = \beta(a, bc)$; (later)
- $A \cong A^\vee$ as left A-modules.

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Similarly, A is called symmetric if it has the following equivalent properties

- $A \cong A^\vee$ as (A, A)-bimodules;
- there exists a nonsingular associative R-bilinear form $\beta\colon A \times A \to R$ that is *symmetric*: $\beta(a, b) = \beta(b, a)$.

Nonsingularity

The R-bilinear form $\beta\colon A \times A \to R$ is said to be nonsingular if it satisfies the following equivalent conditions:

- the map $A \to A^\vee$, $a \mapsto \beta(\,.\,, a)$, is an isomorphism;
- there exist dual bases $\{x_i\}_1^n \subseteq A$ and $\{y_i\}_1^n \subseteq A$ satisfying

$$a = \sum_i \beta(a, y_i) x_i \quad \text{for all } a \in A.$$

These conditions are actually left-right symmetric, even if β is not symmetric.

Nonsingularity

Remark: Symmetry and the Frobenius property are stable under base change $R \to R'$:

If A is a Frobenius or symmetric then the R'-algebra $A \otimes_R R'$ is likewise, with form $\beta \otimes 1_{R'}$ and dual bases $\{x_i \otimes 1_{R'}\}$, $\{y_i \otimes 1_{R'}\}$.

Separability locus

Goal: For a given Frobenius algebra A, determine

$$\{\mathfrak{p} \in \operatorname{Spec} R \mid A \otimes_R Q(R/\mathfrak{p}) \text{ is separable}\}$$

This is equivalent to $A \otimes_R R_{\mathfrak{p}}$ being separable.

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We use techniques and results from

Donald G. Higman, *On orders in separable algebras*, Canad. J. Math. **7** (1955), 509–515

Separability locus

For a fixed nonsingular associative R-bilinear form $\beta\colon A \times A \to R$, define the Casimir operator

$$c = c_\beta\colon A \to \mathcal{Z}(A)\ , \quad a \mapsto \sum_i y_i a x_i$$

Here $\{x_i\}_1^n, \{y_i\}_1^n \subseteq A$ are dual bases for β.

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One checks:

- c is independent of the choice of dual bases $\{x_i\}_1^n, \{y_i\}_1^n$;
- $c(A)$ is an ideal of $\mathcal{Z}(A)$ which is independent of the choice of β ("Casimir ideal").

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Of particular importance will be the "Casimir element"

for A symmetric

$$z = z_\beta = c_\beta(1) = \sum_i y_i x_i = \sum_i x_i y_i \in \mathcal{Z}(A)$$

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Here $\{x_i\}_1^n, \{y_i\}_1^n \subseteq A$ are dual bases for β.

Thm 1 $A \otimes_R Q(R/\mathfrak{p})$ *is separable* $\iff$ $\mathfrak{p} \not\supseteq c(A) \cap R$.

Characters and integrality

Notation:

B	is a fin. dim'l semisimple algebra / field $\Bbbk$
M	an absolutely irreducible B-module
χ_M	the character of M: $\chi_M(b) = \mathrm{Tr}(b_M)$
ω_M	the central character, $\omega_M : \mathcal{Z}(B) \to \mathrm{End}_B(M) = \Bbbk$
$e(M)$	the primitive idempotent $\in \mathcal{Z}(B)$ corresponding to M

Characters and integrality

R**-form**:

$R \subseteq \Bbbk$	is a commutative subring
$A \subseteq B$	a symmetric R-subalgebra of B with $B = A \otimes_R \Bbbk$
$\{x_i\}, \{y_i\}$	dual bases of A for some fixed symmetric nonsing. assoc. bilinear form $\beta\colon A \times A \to R$

Characters and integrality

Theorem 2 (Curtis, T. Fossum, . . . late '60s) *The element*

$$z(M) = z_\beta(M) = \sum_i \chi_M(x_i) y_i \in B$$

is independent of the choice of $\{x_i\}, \{y_i\}$ *and belongs to* $\mathcal{Z}(B)$. *Moreover,*

(a) $\omega_M(z(M)) \in \Bbbk$ *is nonzero and integral over* R;

(b) $e(M) = \omega_M(z(M))^{-1} z(M)$;

(c) $\chi_M(z) = \chi_M(z(M)) \dim_\Bbbk(M)$. *Consequently, if* $\dim_\Bbbk M$ *is nonzero in* $\Bbbk$ *then* $\omega_M(z(M)) = \frac{\omega_M(z)}{\dim_\Bbbk M \cdot 1_\Bbbk}$.

Casimir element

Applications

Hopf algebras

To start with, we consider Hopf algebras over R:

$$H = (H, u, m, \varepsilon, \Delta, \mathrm{S})$$

Hopf algebras

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$$H = (H, u, m, \varepsilon, \Delta, \mathrm{S})$$

Recall that the R-module of right integrals in H is defined by

$$\int_H^r = \{t \in H \mid th = \varepsilon(h)t\}$$

Remark: If H is R-finite then S is bijective and $\int_H^l = \mathrm{S}(\int_H^r)$.

Theorem 3 (Larson-Sweedler ’69, Pareigis ’71, Oberst-Schneider ’73)

(a) H *is a Frobenius* R*-algebra if and only if*

(i) H *is finite over* R*, and*

(ii) $\int_H^r \cong R$.

In particular, this holds if H *is finite over* R *and* $\operatorname{Pic} R = 1$.

(b) *Assume* H *is Frobenius. Then* H *is symmetric iff*

(i) H *is unimodular (i.e.,* $\int_H^l = \int_H^r$*), and*

(ii) S^2 *is an inner automorphism of* H.

Assume H Frobenius and fix a generator $\Lambda \in \int_H^r$; so

$$\int_H^r = R \cdot \Lambda$$

bilinear form: There is a unique $f \in H^\vee$ with $f\Lambda = \varepsilon$.

$$\beta(h,k) = f(hk)$$

dual bases: $\{x_i\} = \{\Lambda_2\}$, $\{y_i\} = \{\mathrm{S}(\Lambda_1)\}$ $(\Delta(\Lambda) = \sum \Lambda_1 \otimes_R \Lambda_2)$

Application #1: separability locus of H

The **Casimir operator** is given by the right adjoint action of Λ on H:

$$c = c_\Lambda \colon H \to \mathcal{Z}(H)\ , \quad h \mapsto \sum \mathrm{S}(\Lambda_1) h \Lambda_2$$

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Thus, $R\varepsilon(\Lambda) = \varepsilon(\int_H^r) \subseteq c(H) \cap R$. In fact, equality holds:

$$c(H) \cap R \subseteq \varepsilon(c(H)) = R\varepsilon(\Lambda)$$

Application #1: separability locus of H

Theorem 1 now gives the following version of a classical result due to Larson and Sweedler:

Corollary 1 *The separability locus of a Frobenius Hopf algebra H over R is*

$$\operatorname{Spec} R \setminus V(\varepsilon(\textstyle\int_H^r))$$

Grothendieck rings

From now on:

$\Bbbk$	is an alg. closed field, $\operatorname{char}\Bbbk = 0$
H	is a semisimple Hopf algebra / $\Bbbk$
H-mod	is the category of fin. gen. left H-modules, a finite tensor category
$\operatorname{Irr} H$	is a full set of irreducible H-modules

Grothendieck rings

The **Grothendieck ring** $G_0(H) = K_0(H\text{-}\,\mathsf{mod})$ is a symmetric $\mathbb{Z}$-algebra.

bilinear form: $\beta([V],[W]) = \dim_{\Bbbk} \operatorname{Hom}_H(V, W^*)$

$\Bbbk$-linear dual

dual bases: $\{[V] \mid V \in \operatorname{Irr} H\}, \{[V^*] \mid V \in \operatorname{Irr} H\}$

Grothendieck rings

The **Grothendieck ring** $G_0(H) = K_0(H\text{-mod})$ is a symmetric $\mathbb{Z}$-algebra.

Casimir element:

$$z = \sum_{V \in \operatorname{Irr} H} [V^*][V] = [H_{\mathrm{ad}}] \ ,$$

the class of $\mathrm{ad} = \mathrm{ad}_l \colon H \to \mathrm{End}_{\Bbbk}(H)$, $\mathrm{ad}(h)(k) = \sum h_1 k \,\mathrm{S}(h_2)$.

Application #2: separability locus of $G_0(H)$

By Thm 1, the issue is to determine $c(G_0(H)) \cap \mathbb{Z}$ for

$$c\colon G_0(H) \to \mathcal{Z}G_0(H), \quad [M] \mapsto \sum_{V \in \operatorname{Irr} H} [V^*][M][V] \ .$$

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the Casimir operator.

Unfortunately, I don't know the answer in general.
Here is what I know ...

Application #2: separability locus of $G_0(H)$

Theorem 4

(a) *If* p *divides* $\dim_{\Bbbk} H$ *then* $G_0(H) \otimes_{\mathbb{Z}} \mathbb{F}_p$ *is not semisimple.*

(b) $G_0(H) \otimes_{\mathbb{Z}} \mathbb{F}_p$ *is semisimple for all* $p > \dim_{\Bbbk} H$*.*

(c) *If* $G_0(H)$ *is commutative then* $G_0(H) \otimes_{\mathbb{Z}} \mathbb{F}_p$ *is semisimple if and only if* p *does not divide* $\dim_{\Bbbk} H$*.*

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(b) *$G_0(H) \otimes_{\mathbb{Z}} \mathbb{F}_p$ is semisimple for all $p > \dim_{\Bbbk} H$.*

(c) *If $G_0(H)$ is commutative then $G_0(H) \otimes_{\mathbb{Z}} \mathbb{F}_p$ is semisimple if and only if p does not divide $\dim_{\Bbbk} H$.*

Part (a) follows from the augmentation $\dim = \dim_{\Bbbk} \colon G_0(H) \to \mathbb{Z}$

$$(\dim_{\Bbbk} H) = (\dim z) = \dim c(G_0(H)) \supseteq c(G_0(H)) \cap \mathbb{Z} \ .$$

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(b) *$G_0(H) \otimes_{\mathbb{Z}} \mathbb{F}_p$ is semisimple for all $p > \dim_{\Bbbk} H$.*

(c) *If $G_0(H)$ is commutative then $G_0(H) \otimes_{\mathbb{Z}} \mathbb{F}_p$ is semisimple if and only if p does not divide $\dim_{\Bbbk} H$.*

For (b) and (c), I consider the norm of the adjoint class

$$z = c(1) = [H_{\mathrm{ad}}] \ .$$

The character algebra

The *character map*

$$\chi \colon G_0(H) \longrightarrow H^* \ , \qquad [V] \mapsto \chi_V$$

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$$\chi\colon G_0(H) \to H^* \,, \qquad [V] \mapsto \chi_V$$

is a ring embedding. One identifies $G_0(H) \otimes_{\mathbb{Z}} \Bbbk$ with the *character algebra* of H,

$$R(H) = \bigoplus_{V \in \operatorname{Irr} H} \Bbbk \chi_V \subseteq H^*$$

This is a semisimple $\Bbbk$-algebra (Theorem 4) with $\mathbb{Z}$-form $G_0(H)$.

The character algebra

The *character map*

$$\chi\colon G_0(H) \to H^* \,, \qquad [V] \mapsto \chi_V$$

is a ring embedding. The embedding respects augmentations,

$$\begin{array}{ccc} G_0(H) & \xhookrightarrow{\ \chi\ } & H^* \\ {\scriptstyle \dim}\downarrow & & \downarrow{\scriptstyle u^*} \\ \mathbb{Z} & \hookrightarrow & \Bbbk \end{array}$$

and involutions:

$$\chi_{V^*} = \mathrm{S}^*(\chi_V)$$

Application #3: the class equation

Theorem 5 (G.I. Kac '72, Y. Zhu '94)

$\dim_{\Bbbk} H^* e$ *divides* $\dim_{\Bbbk} H^*$ *for every primitive idempotent* $e = e^2 \in R(H)$.

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Taking $M = R(H)e$, this is an application of Theorem 2:

$$\frac{\omega_M([H_{\mathrm{ad}}])}{\dim_{\Bbbk} M} \quad \text{is integral over } \mathbb{Z}.$$

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Remarks:

(1) With $H = \Bbbk G$ for G a finite group, this becomes:

The sizes of all conjugacy classes of G divide $|G|$.

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Remarks:

(2) With $H = (\Bbbk G)^*$ one obtains **Frobenius' Theorem**:

The dimensions of simple $\Bbbk G$-modules divide $|G|$.

Application #3: the class equation

Theorem 5 (G.I. Kac '72, Y. Zhu '94)

$\dim_{\Bbbk} H^*e$ *divides* $\dim_{\Bbbk} H^*$ *for every primitive idempotent* $e = e^2 \in R(H)$.

Remarks:

(3) Theorem 5 is used in the proof of Theorem 4.

A problem

Adjoint representation and Chevalley property

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G	is a finite group

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