Torus actions on affine PI-algebras (M. Lorenz)

R an algebra / field $\Bbbk = \overline{\Bbbk}$

G an affine alg. $\Bbbk$-group acting rationally by $\Bbbk$-alg. auto's on R

Assume G connected, for simplicity

① Background (L., Transf. Groups 2009)

$$\begin{aligned} \operatorname{Spec} R &\twoheadrightarrow G\text{-}\operatorname{Spec} R := \{G\text{-primes of } R\} \\ &\qquad\qquad = \{G\text{-stable primes}\} \\ P &\longmapsto \bigcap_{g \in G} g(P) \end{aligned}$$

Main Thm in [L]: a description of the fibres of this map in terms of explicit commutative spectra.

Heuristic fact: In numerous examples of quantized coordinate algebras, there is a natural action of some alg. torus G with

$$G\text{-}\operatorname{Spec} R \text{ finite (and interesting!)}$$

But: no general finiteness criterion is known!

$\hookrightarrow$? Problem in Brown–Goodearl

Except . . .

$\left\{ \begin{array}{l} R \text{ a Jacobson ring \&} \\ \mathcal{Z}(\mathrm{End}_R M) = \Bbbk \ \forall M \in \mathrm{Irr}\, R \end{array} \right.$

$\boxed{\text{Prop}^n}$ Assume that R sat^s the Nullstellensatz. Then:

G-Spec R is finite $\Longleftrightarrow$ R sat^s

(i) ACC on G-stable semiprime ideals

(ii) the Dixmier-Moeglin equivalence (see below)

(iii) G-Spec R = G-Rat R

$\Longleftrightarrow$ $G \backslash \mathrm{Rat}\, R$ is finite

Here

- $P \in$ G-Spec R is called G-rational if $\mathcal{C}(R/P)^G = \Bbbk$.

"extended centroid" (see later)

rational $\equiv$ 1-rational

- Dixmier-Moeglin equivalence: For all $P \in \mathrm{Spec}\, R$,

loc. closed in Spec R $\equiv$ primitive $\equiv$ rational (1-rational)

topology — repr. th^y — ring th^y

2 Affine PI-algebras

From now on R affine PI-algebra $/ \Bbbk$

commutative

e.g. $R = M_n(\Bbbk[x_1, \ldots, x_m])$

sat^s $\sum_{\sigma \in S_{2n}} \operatorname{sgn}(\sigma)\, y_{\sigma(1)} \cdots y_{\sigma(2n)}$

(Amitsur-Levitzki Thm)

Some basic properties (Kaplansky, Amitsur, Procesi, ...):

- R has ACC on all semiprime ideals ✓
- R sat^s the Nullstellensatz & the Dixmier–Moeglin equivalence; in fact

  max'l ≡ loc. closed ≡ primitive ≡ rat'l ≡ fin. codim'l

- Assume R prime. Put $Z = \mathfrak{z}(R)$ & $F = Q(Z)$.

  Then $Q(R) := R[(Z \setminus 0)^{-1}]$ is a central simple F-algebra.

$\rightsquigarrow$ Def^n $\operatorname{PIdeg} R := (\dim_F Q(R))^{1/2}$

- In general, put

$$I_d = \bigcap \{ P \in \operatorname{Spec} R \mid \operatorname{PIdeg} R/P \leq d \}$$

to get a finite (!) chain of semiprime ideals of R

$I_n \subseteq I_{n-1} \subseteq \ldots \subseteq I_1$. Put

$$\boxed{\mathcal{S} := \{ P \in \operatorname{Spec} R \mid P \text{ is min'l over some } I_d \}}$$

a finite (!) subset of $\operatorname{Spec} R$.

③ Torus actions on R

Back to G-actions ...

Note that all I_d are G-stable, and hence so are all $P \in \mathcal{S}$:

$$\mathcal{S} \subseteq G\text{-}\operatorname{Spec} R.$$

?

Recall:

- $G\text{-}\operatorname{Spec} R$ is finite $\iff$ $G\text{-}\operatorname{Spec} R = G\text{-}\operatorname{Rat} R$
- $P \in G\text{-}\operatorname{Spec} R$ is G-rational

$$\overset{\text{def}}{\iff} \quad \mathcal{C}(R/P)^G = \Bbbk$$

for R PI:

$$= Q(\mathcal{Z}(R/P))^G$$

Now assume that $G \cong (\Bbbk^\times)^r$ is an <u>alg. torus</u>.

Then:

- $P \in G\text{-Rat}\,R \overset{\text{easy}}{\iff} \dim_{\Bbbk} \mathcal{Z}(R/P)_\lambda \leq 1$

  for all $(\lambda : G \to \Bbbk^\times) \in X(G)$

Here $M_\lambda = \{ m \in M \mid gm = \lambda(g)m \ \forall g \in G\}$ for any G-module M. The torus G acts <u>rationally</u> on R iff there are algebra generators for R that belong to various R_λ.

Here is the main result of this talk (not written up yet) ...

$\boxed{\text{Thm}}$ Let R, G and $\mathcal{P} \in G\text{-Spec}\,R$ be as above. Then the following are equivalent:

(i) $G\text{-Spec}\,R$ is finite;

(ii) $\mathcal{P} \in G\text{-Rat}\,R$;

(iii) $P \in \mathcal{P} \Rightarrow \dim_{\Bbbk} \mathcal{Z}(R/P)_\lambda \leq 1 \ \forall \lambda \in X(G)$.

Classical example.

(Ref: H. Kraft, Geom. Methoden in der Invariantenth.)
p. 105

R = affine commutative domain
X = MaxSpec R = Rat R

$\rightsquigarrow \mathcal{Y} = \{(0)\}$

G-Spec R is finite $\overset{\text{Thm}}{\Longleftrightarrow}$ $\mathbb{k}(X)^G = \mathbb{k}$; so (0) is rational

Prop^n $\Updownarrow$ $\Uparrow$ clear

$G\backslash X$ is finite $\underset{\text{clear}}{\Longrightarrow}$ $\exists$ dense G-orbit in X

Thus all are equivalent.