

Torus actions on affine PI-algebras (M. Lorenz)

R an algebra / field $k = \bar{k}$

G an affine alg. k -group acting rationally by k -alg. auto's on R

Assume G connected, for simplicity

① Background (L., Transf. Groups 2009)

$$\text{Spec } R \longrightarrow G\text{-Spec } R := \{G\text{-primes of } R\} \\ = \{G\text{-stable primes}\}$$

$$\bigcup_{\mathcal{P}} \mathcal{P} \longrightarrow \bigcap_{g \in G} g(\mathcal{P})$$

Main Thm in [L.]: a description of the fibres of this map in terms of explicit commutative spectra.

Heuristic fact: In numerous examples of quantized coordinate algebras, there is a natural action of some alg. torus G with

$G\text{-Spec } R$ finite (and interesting!)

But: no general finiteness criterion is known!



Problem in
Brown - Goodearl

R a Jacobson ring &
 $Z(\text{End}_R M) = k \quad \forall M \in \text{Ir-}R$

Propⁿ Assume that R sat^s the Nullstellensatz. Then:

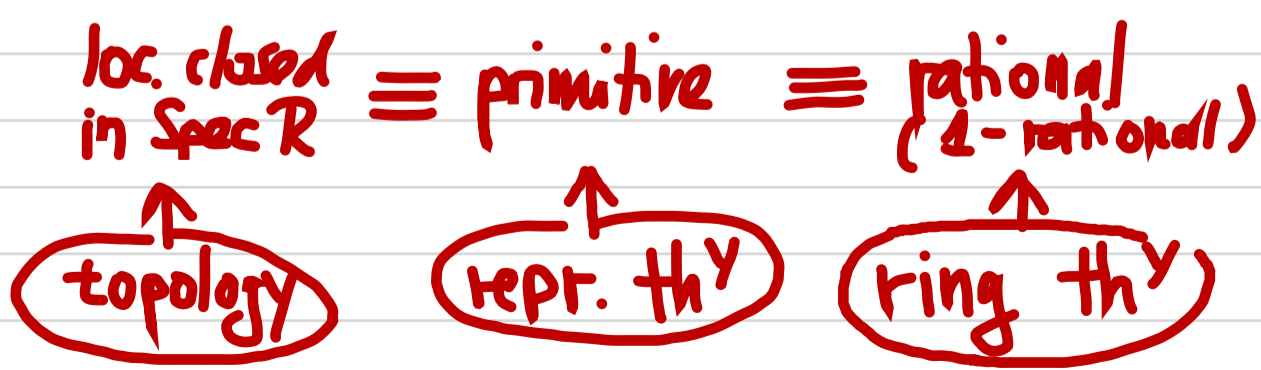
- $G\text{-Spec } R \text{ is finite} \iff R \text{ sat}^s$
- (i) ACC on G -stable semiprime ideals
 - (ii) the Dixmier-Moeglin equivalence (see below)
 - (iii) $G\text{-Spec } R = G\text{-Rat } R$

$\iff G \setminus \text{Rat } R \text{ is finite}$

Here \bullet $P \in G\text{-Spec } R$ is called G -rational if $e(R/P)^G = k$.
 "extended centroid" (see later)

rational \equiv 1-rational

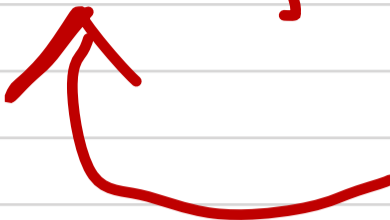
Dixmier-Moeglin equivalence: For all $P \in \text{Spec } R$,



② Affine PI-algebras

From now on R affine PI-algebra / k

Commutative



e.g. $R = M_n(k[x_1, \dots, x_m])$
 sat^s $\sum_{\sigma \in S_{2n}} \text{sgn}(\sigma) y_{\sigma(1)} \cdots y_{\sigma(2n)}$
 (Amitsur-Lovitzki Thm)

Some basic properties (Kaplansky, Amitsur, Procesi, ...):

- R has ACC on all semiprime ideals ✓
- R sat^s the Nullstellensatz & the Dixmier-Moeglin equivalence; in fact

max^l \equiv loc. closed \equiv primitive \equiv rat^l \equiv fin. codim^l

- Assume R prime. Put $Z = Z(R)$ & $F = Q(Z)$.

Then $Q(R) := R[(Z \setminus 0)^{-1}]$ is a central simple

F -algebra.

\leadsto Defⁿ $\text{PIdeg } R := (\dim_F Q(R))^{1/2}$

- In general, put

$$I_d = \bigcap \{ P \in \text{Spec } R \mid \text{PIdeg } R/P \leq d \}$$

to get a finite (!) chain of semiprime ideals of R

$$I_n \subseteq I_{n-1} \subseteq \dots \subseteq I_1. \text{ Put}$$

$$\mathcal{Y} := \{ P \in \text{Spec } R \mid P \text{ is min'l over some } I_d \}$$

a finite (!) subset of $\text{Spec } R$.

③ Torus actions on R

Back to G -actions ...

Note that all I_d are G -stable, and hence so are all

$P \in \mathcal{Y}$:

$$\mathcal{Y} \subseteq G\text{-Spec } R.$$

?

Recall:

- $G\text{-Spec } R$ is finite $\Leftrightarrow G\text{-Spec } R = G\text{-Rat } R$

- $P \in G\text{-Spec } R$ is G -rational

\Leftrightarrow
def

$$\mathcal{Q}(R/P)^G = k$$

for $R \text{ PI} :$
 $= \mathcal{Q}(\mathcal{Z}(R/P))^G$

Now assume that $G \cong (k^\times)^r$ is an alg. torus.

Then:

- $P \in G\text{-Rat } R \stackrel{\text{easy}}{\iff} \dim_{k^*} \mathfrak{z}(R/P)_\lambda \leq 1$
for all $(\lambda: G \rightarrow k^\times) \in X(G)$

Here $M_\lambda = \{m \in M \mid gm = \lambda(g)m \ \forall g \in G\}$ for any G -module M . The torus G acts rationally on R iff there are algebra generators for R that belong to various R_λ .

Here is the main result of this talk (not written up yet) ...

Thm Let R, G and $\mathcal{P} \subseteq G\text{-Spec } R$ be as above.

Then the following are equivalent:

- $G\text{-Spec } R$ is finite;
- $\mathcal{P} \subseteq G\text{-Rat } R$;
- $P \in \mathcal{P} \implies \dim_{k^*} \mathfrak{z}(R/P)_\lambda \leq 1 \ \forall \lambda \in X(G)$.

Classical example.

(Ref: H. Kraft, Geom. Methoden in der Invariantenthe.)
p. 105

$R =$ affine commutative domain

$X = \text{MaxSpec } R = \text{Rat } R$

$\leadsto \mathcal{Y} = \{(0)\}$

$G\text{-Spec } R$
is finite

Thm
 \Leftrightarrow

$k(X)^G = k$; so
 (0) is rational

Propⁿ \Uparrow

\Uparrow clear

$G \backslash X$ is finite $\xRightarrow{\text{clear}}$ \exists dense G -orbit in X

Thus all are equivalent.