MULTIPLICATIVE INVARIANTS AND SEMIGROUP ALGEBRAS

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ABSTRACT. Let G be a finite group acting by automorphism on a lattice A, and hence on the group algebra $S = k[A]$. The algebra of G-invariants in S is called an algebra of multiplicative invariants. We present an explicit version of a result of Farkas stating that multiplicative invariants of finite reflection groups are semigroup algebras.

INTRODUCTION

This article continues our investigaton of multiplicative invariants in [12, 13, 14] and is motivated by Farkas' work in [3, 4, 5].

Our specific focus here is a suitable permanence theorem for multiplicative actions of finite groups analogous to the classical Shephard-Todd-Chevalley Theorem for "linear" actions of finite groups (of good order) on polynomial algebras; this theorem states precisely when the corresponding algebra of invariants is again a polynomial algebra (e.g., [1, p. 115]).

Multiplicative actions, also called exponential actions [1], are certain group actions on Laurent polynomial rings or, equivalently, group algebras of lattices. Specifically, let A denote a lattice, i.e., a free abelian group of finite rank, and let G be a group acting by automorphisms on A. This action extends uniquely to an action of G on the group algebra $S = k[A]$ of A. Actions of this type are referred to as *multiplicative actions*, and the resulting algebra of invariants $R = S^G$ is called an algebra of *multiplicative invariants*. It is easy to see that R is again a group algebra only if G acts trivially on A; see (1.3). Thus the permanence theorem we have in mind is a characterization of all multiplicative actions yielding invariants that are semigroup algebras.

Here is the state of affairs and our contribution. It is implicit in [4, proof of Theorem 10] that multiplicative invariants of finite reflection groups are indeed semigroup algebras; this has been pointed out by Farkas himself in [5, p. 72]. Related work appears in [16].

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After deploying the requisite background material and some technicalities in Section 1, we present in Section 2 an explicit proof of Farkas' theorem, working over an arbitrary commutative base ring k, along with an analysis of the structure of the corresponding semigroup and of the class group of the invariant algebra. The result, Theorem (2.4), is derived from a classical fact [1, Théorème 1 on p. 188] concerning multiplicative invariants of Weyl group actions on weight lattices of root systems. The method employed leads directly to an explicit fundamental system of invariants.

I don't know if the converse of Theorem (2.4) holds: Do all multiplicative invariants that are semigroup algebras come from reflection groups? If G acts fixed point freely on A/A^G and A/A^G has rank at least 2, the invariant algebra R will never be a semigroup algebra. In particular, this holds for all multiplicative actions of finite groups of odd prime order. The proof of this result, based on an investigation of the singularities of multiplicative invariants, is not included in the present article, as doubtless a good deal more can be said. I hope to return to this question in a future publication.

Notations and Conventions. Throughout this note, k will denote a commutative ring (with $1 \neq 0$) unless explicitly noted otherwise. All monoids considered in this article are understood to be commutative. We use $\mathbb{Z}_+$ to denote the set of nonnegative integers. Further notation will be introduced below, in particular in (1.3).

1. PRELIMINARIES

1.1. **Commutative semigroup algebras.** Let M denote a monoid, with operation written as multiplication and identity element 1, and let $k[M]$ denote the semigroup (or monoid) algebra of M over k. Thus every element $\alpha \in k[M]$ can be uniquely written as a finite linear combination

$$\alpha = \sum_{m \in M} k_m m \quad \text{with } k_m \in k \; .$$

The set $\operatorname{Supp}(\alpha) = \{m \in M \mid k_m \neq 0\}$ is called the *support* of α. Multiplication in $k[M]$ is defined by k-linear extension of the multiplication of M.

A good reference for general ring theoretic properties of commutative semigroup algebras is [6]. We note in particular the following facts:

- The k-algebra $k[M]$ is finitely generated (*affine*) if and only if M is a finitely generated monoid. This is trivial.
- $k[M]$ is a domain iff k is a domain and M is *cancellative* ($ax = ay \Rightarrow x = y$ for $a, x, y \in M$) and *torsion-free* ($x^n = y^n, n > 0 \Rightarrow x = y$ for $x, y \in M$); see [6, Theorem 8.1].
- Assume $k[M]$ is a domain. Then $k[M]$ is integrally closed iff k is integrally closed and M is *normal*: $x^n = y^n z$ for $x, y, z \in M$ implies $z = z_1^n$ for some $z_1 \in M$; see [6, Corollary 12.11].

1.2. **Affine normal semigroups.** Finitely generated cancellative torsion-free normal monoids are often simply referred to as *affine normal semigroups*. By (1.1), we have for any monoid M:

The k-algebra $k[M]$ is an affine integrally closed domain iff k is an integrally closed domain and M is an affine normal semigroup.

As a reference for affine semigroup algebras in particular, I recommend [2]. By [2, Proposition 6.1.3], affine normal semigroups M have the following structure:

$M = \mathrm{U}(M) \times M_+$, where $\mathrm{U}(M)$, the group of units of M, is a free abelian group of finite rank and M_+ is an affine normal semigroup that is positive, *that is,* $\mathrm{U}(M_+) = \{1\}$.

If M is affine normal and k a domain, the group of units of $k[M]$ is given by:

$$\mathrm{U}(k[M]) = \mathrm{U}(k) \times \mathrm{U}(M)\ ; \tag{1}$$

see [6, Theorem 11.1]. The k-algebra map $\mu : k[M] \to k$ that is given by $\mu(m) = 1$ for $m \in \mathrm{U}(M)$ and $\mu(m) = 0$ if $1 \neq m \in M_+$ is called the *distinguished augmentation* of $k[M]$.

1.3. **Multiplicative Invariants.** The following **notations** will be kept throughout this article:

A	will be a free abelian group of finite rank;
$S = k[A]$	will denote the group algebra of A over k;
G	will be a finite group acting be automorphisms on A, and hence on S as well; the action will be written exponentially, $a \mapsto a^g$;
$R = S^G$	is the subalgebra of G-invariants in S.

In this situation, A is often called a *G-lattice*. As our main concern is R, the algebra of multiplicative G-invariants, we may assume that the G-lattice A is *faithful*, that is, the map $G \to \mathrm{GL}(A)$ that defines the G-action is injective. Finally, A will be called *effective* if the subgroup A^G of G-invariant elements of A is trivial.

The *orbit sum* of an element $a \in A$ is the element of S that is defined by

$$\sigma(a) = \sum_{x \in a^G} x \ \in S\ .$$

where $a^G = \{a^g \mid g \in G\} \subseteq A$ denotes the G-orbit of a. Orbit sums are clearly G-invariant, and hence they actually belong to R. In fact, they provide a k-basis for R :

$$R = \bigoplus_{a \in A/G} k\sigma(a)\ ,$$

where A/G denotes a transversal for the G-orbits in A. The structure constants for this basis belong to the subring k_0 of k that is generated by 1. Thus, as a k-algebra, R is defined over k_0: $R = k \otimes_{k_0} R_0$ with $R_0 = \bigoplus_{a \in A/G} k_0\sigma(a) = k_0[A]^G$. In particular, R is an affine k-algebra, because R_0 is affine over k_0 by Noether's theorem. Moreover, if k is a domain or integrally closed then R is likewise, since both properties pass from k to S (A is an affine normal semigroup) and from S to the invariant subalgebra R.

Remark. Substantiating a remark made in the introduction, we claim that if R is a group algebra over k then G acts trivially on A. Indeed, we may assume k to be a field by fixing a map $k \to K$ into a field K and noting that $K \otimes_k R = K[A]^G$. Since group algebras are generated, as k-algebras, by units and $\mathrm{U}(R) = \mathrm{U}(S)^G = k^* \times A^G$, by (1.2)(1), we conclude that $R = k[A^G]$. Now S is integral over $R = S^G$ and, on the other hand, A/A^G is torsion-free. Thus we must have $A = A^G$, as desired.

1.4. **Passage to an effective lattice.** Let $^-$ denote the canonical map $A \twoheadrightarrow A/A^G$ and its extension to S; so

$$^- : S = k[A] \twoheadrightarrow \overline{S} = k[A/A^G] , \quad a \mapsto aA^G \ (a \in A) .$$

Note that $\overline{A} = A/A^G$ is a G-lattice and the map $^-$ is G-equivariant. Moreover, letting G_x denote the isotropy (stabilizer) subgroup of G of an element x in A or in $\overline{A}$, we have

$$G_a = G_{\overline{a}} \quad \text{for all } a \in A. \tag{2}$$

Here, the inclusion $G_a \subseteq G_{\overline{a}}$ is clear. The reverse inclusion follows from the fact that the map $G_{\overline{a}} \to A^G$, $g \mapsto a^g a^{-1}$, is a group homomorphism, and hence it must be trivial, as $G_{\overline{a}}$ is finite while A^G is torsion free. We deduce from the above equality of isotropy groups that

$\overline{A}$ is an effective G-lattice.

Further, $^- : S \to \overline{S}$ sends the orbit sum $\sigma(a)$ to the orbit sum $\sigma(\overline{a})$, and $\sigma(\overline{a}) = \sigma(\overline{b})$ is equivalent to $\sigma(a) = \sigma(b)c$ for some $c \in A^G$. Consequently,

The map $^-$ maps R onto the G-invariants in $\overline{S}$, that is, $\overline{R} = \overline{S}^G$. The kernel of this epimorphism is the ideal $\big(a - 1 \mid a \in A^G\big)$ of R.

Finally, every G-equivariant homomorphism from A to some effective G-lattice clearly factors through $^- : A \to \overline{A}$.

2. Reflection Groups

2.1. **Reflections.** An endomorphism ϕ of a vector space is called a *pseudoreflection* if $\mathrm{Id} - \phi$ has rank 1; ϕ is a *reflection* if, in addition, $\phi^2 = \mathrm{Id}$.

Keeping the notation of (1.3), we will assume in this section that A is a G-lattice which, without essential loss, will be assumed faithful. We will further assume that G is a reflection group on A; so:

G is a finite subgroup of $\mathrm{GL}(A)$ that is generated by reflections.

Here, an element $g \in G$ is called a *reflection* if g is a reflection on $A \otimes_{\mathbb{Z}} \mathbb{Q}$. We remark that, since $\det g = \pm 1$ holds for all $g \in G$, pseudoreflections in G are automatically reflections. They can also be characterized by the condition that the subgroup $A^{\langle g \rangle} = \mathrm{Ker}_A(g - \mathrm{Id})$ of g-fixed points in A have rank equal to $\mathrm{rank}(A) - 1$ or, alternatively, $g^2 = \mathrm{Id}$ and

$\mathrm{Ker}_A(g + \mathrm{Id}) = \{a \in A \mid a^g = a^{-1}\}$ *is infinite cyclic.*

As in (1.4), we let $\overline{}$ denote the canonical map $A \twoheadrightarrow \overline{A} = A/A^G$. Note that (1.4)(2) implies that $\overline{A^{\langle g \rangle}} = \overline{A}^{\langle g \rangle}$ holds for all $g \in G$. Therefore, if g acts as a reflection on A then it does so on $\overline{A}$ as well, and conversely.

2.2. **Root systems.** Embed A into the $\mathbb{R}$-vector space $V = A \otimes_{\mathbb{Z}} \mathbb{R}$ and view G as a subgroup of $\mathrm{GL}(V)$. As is customary, we will use additive notation in A and V. Define

$$\rho(v) = |G|^{-1} \sum_{g \in G} v^g \qquad (v \in V) .$$

Thus, ρ is an idempotent $\mathbb{R}[G]$-endomorphism of V with $\rho(V) = V^G$, the subspace of G-fixed points in V. Putting $\pi = 1 - \rho \in \mathrm{End}_{\mathbb{R}[G]}(V)$, we obtain

$$A \subseteq \rho(A) \oplus \pi(A) \subseteq \rho(V) \oplus \pi(V) = V .$$

For each reflection $g \in G$, let the two possible generators of $\mathrm{Ker}_A(g + \mathrm{Id})$ be denoted $\pm a_g$. Define

$$\Phi = \Phi_{A,G} = \{\pm a_g \mid g \text{ a reflection in } G\} .$$

The crucial properties of Φ are listed in the following lemma; see Farkas [4, Lemmas 1–3].

Lemma. *$\Phi = \Phi_{A,G}$ is a reduced crystallographic root system in $\pi(V)$ and the restriction of G to $\pi(V)$ is the Weyl group $\mathcal{W}(\Phi)$ of Φ. Furthermore,*

$$\mathbb{Z}\Phi \subseteq A \subseteq \pi^{-1}(\Lambda) ,$$

where $\mathbb{Z}\Phi$, the $\mathbb{Z}$-span of Φ in V, is the root lattice *and $\Lambda = \Lambda_{A,G} = \{v \in \pi(V) \mid v - v^g \in \mathbb{Z}a_g$ for all reflections $g \in G\}$ is the* weight lattice *of Φ.*

For background on root systems, we refer to [1] or [10].

2.3. **A reduction lemma.** In this section, we will prove a technical lemma stating that an algebra of multiplicative invariants is a semigroup algebra provided a closely related one is. Let

$$\mathcal{M}(A)$$

denote the submonoid of $(R, \cdot)$ that is generated by the orbit sums $\sigma(a)$ for $a \in A$, and similarly for other G-lattices.

Lemma. *Let $A \subseteq B$ be G-lattices such that B/A is G-trivial. Suppose that $k[B]^G = kC$, the k-linear span of some subset $C \subseteq \mathcal{M}(B)$. Then $k[A]^G = kD$ with $D = C \cap k[A]$.*

Proof. Note that D is a subset of $k[A]^G$; so clearly $kD \subseteq k[A]^G$. For the other inclusion, let $\alpha \in k[A]^G$ be given. Then $\alpha = \sum_{c \in C} k_c c$ with $k_c \in k$ almost all zero. We show by induction on the minimum number, $n(\alpha)$, of nonzero terms in such an expression that $\alpha \in kD$. The case $n(\alpha) = 0$ (i.e., $\alpha = 0$) being obvious, assume $\alpha \neq 0$. Then some $d \in C$ with $k_d \neq 0$ must satisfy $\mathrm{Supp}(d) \cap A \neq \emptyset$. Say $d = \sigma(b_1) \cdot \ldots \cdot \sigma(b_l)$ with $b_j \in B$. Then

$$\mathrm{Supp}(d) \subseteq \{b_1^{g_1} \cdot \ldots \cdot b_l^{g_l} \mid g_j \in G\} .$$

So some product $b_1^{g_1} \cdot \ldots \cdot b_l^{g_l}$ belongs to A. Inasmuch as B/A is G-trivial, all these products are congruent to each other modulo A, and hence they all belong to A. Thus, $\mathrm{Supp}(d) \subseteq A$ and so $d \in D$. Since $\alpha - k_d d$ belongs to kD, by induction, we conclude that $\alpha \in kD$ as well. This proves the lemma. □

Note that if the subset C in the Lemma is k-independent or multiplicatively closed then so is $D = C \cap k[A]$. Hence, if $k[B]^G = kC$ is a semigroup algebra, with semigroup basis C, then $k[A]^G = kD$ is a semigroup algebra with semigroup basis D.

We also remark for future use that the argument in the proof of the Lemma shows that, for $d = \prod_{j=1}^{l} \sigma(b_j) \in \mathcal{M}(B)$,

$$\prod_{j=1}^{l} \sigma(b_j) \in k[A] \Longleftrightarrow \mathrm{Supp}(d) \cap A \neq \emptyset \Longleftrightarrow \prod_{j=1}^{l} b_j \in A\,. \tag{3}$$

2.4. **Multiplicative invariants of reflection groups.** Our goal here is to prove the following result implicit in the work of Farkas [4, 5]. We will use the notation of (2.2).

Theorem. *Let A be a free abelian group of finite rank, and let G be a finite subgroup of $\mathrm{GL}(A)$ that is generated by reflections. Then the invariant algebra $R = k[A]^G$ is a semigroup algebra; in fact, $R \cong k[M]$ with $M = A^G \times (\pi(A) \cap \Lambda_+)$, where Λ_+ is the semigroup of dominant weights for some base of the root system $\Phi_{A,G}$.*

Proof. Fix a base $\Delta = \{\alpha_1, \ldots, \alpha_r\}$ for $\Phi = \Phi_{A,G}$, i.e., Δ is a subset of Φ that is an $\mathbb{R}$-basis of $\pi(V)$ and such that $\Phi \subseteq \mathbb{Z}_+\Delta \cup -\mathbb{Z}_+\Delta$. So $\alpha_i = \pm a_{g_i}$ for certain reflections $g_i \in G$, often called *simple reflections*. The *fundamental dominant weights* $\lambda_1, \ldots, \lambda_r$ are determined by $\lambda_i - \lambda_i^{g_j} = \delta_{i,j}\alpha_j$ (Kronecker delta); they form a $\mathbb{Z}$-basis of the weight lattice Λ. The semigroup Λ_+ of dominant weights for Δ is

$$\Lambda_+ = \oplus_{i=1}^r \mathbb{Z}_+\lambda_i\,.$$

It is a classical result [1, Théorème 1 on p. 188] that $k[\Lambda]^G$ is a polynomial algebra, with the orbit sums of the fundamental dominant weights as independent generators. In other words,

> $k[\Lambda]^G = kE$, *with* $E = \langle \sigma(\lambda_1), \ldots, \sigma(\lambda_r)\rangle \cong \Lambda_+$ *a k-independent submonoid of* $\mathcal{M}(\Lambda)$.

Now put $B = \rho(A) \oplus \Lambda$, a G-lattice in V with $A \subseteq B$ and B/A G-trivial. To see the latter, note that A contains $A^G \oplus \mathbb{Z}\Phi$, and $B/(A^G \oplus \mathbb{Z}\Phi) \cong \big(\rho(A)/A^G\big) \oplus (\Lambda/\mathbb{Z}\Phi)$ is G-trivial, since $\rho(A) \subseteq V^G$ and the Weyl group G of Φ acts trivially on the fundamental group $\Lambda/\mathbb{Z}\Phi$ of Φ; cf. [1, p. 167]. Inasmuch as $k[B] = k[\rho(A)] \otimes_k k[\Lambda]$, with $\rho(A) = B^G$, the G-invariants in $k[B]$ are given by $k[B]^G = k[B^G] \otimes_k k[\Lambda]^G$. Thus, using the above description of $k[\Lambda]^G$,

$$k[B]^G = k[B^G] \otimes_k kE = kC \quad \textit{with}\ \ C = B^G \times E\,.$$

Note that C is a k-independent submonoid of $\mathcal{M}(B)$. Lemma (2.3) therefore implies that $k[A]^G = kD$ is a semigroup algebra, with semigroup basis $D = C \cap k[A]$. It remains to verify the description of the monoid given in the theorem. To this end, note that, by (2.3)(3), the isomorphism $B^G \oplus \Lambda_+ \xrightarrow{\cong} B^G \times E = C$ restricts to an isomorphism $M := (B^G \oplus \Lambda_+) \cap A \xrightarrow{\cong} D$. Furthermore, writing $a \in A$ as $a = \rho(a) + \pi(a)$, we see that $a \in M$ if and only if $\pi(a) \in \Lambda_+$. Since $\mathrm{Ker}_A(\pi) = A^G$ and $\overline{A} = A/A^G$ is free, we have $A = A^G \oplus A'$ with $A' \cong \pi(A)$ via π. This decomposition induces a corresponding one for M, because $A^G \subseteq M$; so $M = A^G \oplus (M \cap A')$ and $M \cap A' \cong \pi(A) \cap \Lambda_+$ via π. This completes the proof of the theorem. □

2.5. **Generators.** We now describe how the foregoing leads to an explicit set of *fundamental invariants*, that is, algebra generators for R. Inasmuch as $R \cong k[M]$, this amounts to finding generators for M and tracing them through the isomorphism. As this isomorphism is the identity on $\mathrm{U}(M) = A^G$, we will concentrate on M_+.

2.5.1. *Generators for* $M_+ = \pi(A) \cap \Lambda_+$. Since the semigroup M_+ is positive, it has a unique minimal generating set, the so-called *Hilbert basis* of M_+. Here, in outline, is how to find this Hilbert basis; for complete details and an algorithmic treatment, see [17, Chapter 13].

Recall that $\Lambda_+ = \oplus_{i=1}^r \mathbb{Z}_+ \lambda_i$, where $\lambda_1, \dots, \lambda_r$ are the fundamental dominant weights. These belong to $\pi(A) \otimes \mathbb{Q} \subseteq V$. Hence, there are suitable $0 \neq z_i \in \mathbb{Z}_+$ so that $z_i \lambda_i \in M_+$; we will assume that z_i is chosen minimal and set $m_i = z_i \lambda_i$ for these choices. The subset

$$K = \sum_{i=1}^r [0, m_i] = \{\sum_{i=1}^r t_i m_i \mid 0 \leq t_i \leq 1\}$$

of V is compact (a zonotope), and hence its intersection $K \cap M_+$ with the discrete M_+ is finite. It is easy to see that $K \cap M_+$ generates M_+; the Hilbert basis of M_+ can be found by selecting the indecomposable elements of $K \cap M_+$, that is, the elements $m \in K \cap M_+$ that cannot be written as $m = n + n'$ with $0 \neq n, n' \in K \cap M_+$. Note that $m_1, \dots, m_r$ are certainly indecomposable, by the minimal choice of the z_i's and linear independence of the λ_i's. The remaining indecomposables in $K \cap M_+$ (if any) will be denoted $m_{r+1}, \dots, m_s$; so $s \geq r = \mathrm{rank}(\overline{A})$.

2.5.2. *Fundamental invariants.* As all $m_i \in \Lambda_+ = \oplus_{j=1}^r \mathbb{Z}_+ \lambda_j$, they have a unique representation of the form $m_i = \sum_j z_{i,j} \lambda_j$ with $z_{i,j} \in \mathbb{Z}_+$. For $i \leq r$, this representation is simply $m_i = z_i \lambda_i$, as above. Thus we obtain the following system of fundamental invariants:

$$\mu_i = \prod_{j=1}^r \sigma(\lambda_i)^{z_{i,j}} \ (i = 1, \dots, s)$$

Here, $\mu_1 = \sigma(\lambda_1)^{z_1}, \dots, \mu_r = \sigma(\lambda_r)^{z_r}$ are algebraically independent, as the $\sigma(\lambda_i)$'s are, and R is a finite module over the polynomial algebra $k[\mu_1, \dots, \mu_r]$, since each μ_i, raised to a suitable power, belongs to $\langle \mu_1, \dots, \mu_r \rangle$. In fact, R is a free module over $k[\mu_1, \dots, \mu_r]$, at least if k is a domain. To see this, we may assume that $k = k_0$, as in (1.3); so k is $\mathbb{Z}$ or a finite field. In either case, $R \cong k[M]$ is Cohen-Macaulay, by [9], and freeness follows.

2.6. **The class group.** In this section, we assume that k a factorial domain, for simplicity. The formula given in [12] for the class group of R can be rewritten in terms of the above root system data. Indeed, $R = k[M] = k[A^G] \otimes k[M_+]$ is a Laurent polynomial extension of $k[M_+]$, and so $\mathrm{Cl}(R) = \mathrm{Cl}(k[M_+])$. Further, by (1.4), $k[M_+] \cong \overline{R}$, and by [12], $\mathrm{Cl}(\overline{R}) = \mathrm{H}^1(G, \overline{A}^D)$, where D denotes the subgroup of G that is generated by those reflections that are *diagonalizable* on $\overline{A} = A/A^G$, that is, with respect to a suitable $\mathbb{Z}$-basis of $\overline{A}$, they have the form $\mathrm{diag}(-1, 1, \ldots, 1)$. Now G acts as a reflection group on $\overline{A}^D$, and the G-lattice $\overline{A}^D$ is effective, as $\overline{A}$ is. Thus, [11, Proposition 2.2.25] gives $\mathrm{H}^1(G, \overline{A}^D) \cong \Lambda_{\overline{A}^D, G}/\overline{A}^D$. Hence,

$$\mathrm{Cl}(R) \cong \Lambda_{\overline{A}^D, G}/\overline{A}^D \ .$$

It is perhaps worth noting that $\Lambda_{\overline{A}^D, G}/\overline{A}^D$ is always a direct summand of $\Lambda_{A,G}/\pi(A) = \Lambda_{\overline{A},G}/\overline{A}$. This follows from the fact that $\overline{A}^D$ is a direct summand of $\overline{A}$ as a G-lattice; see [12, Lemma 2.4].

In the special case where A is effective at the outset and G contains no diagonalizable reflections, the above formula simplifies to

$$\mathrm{Cl}(R) \cong \Lambda/A \ ,$$

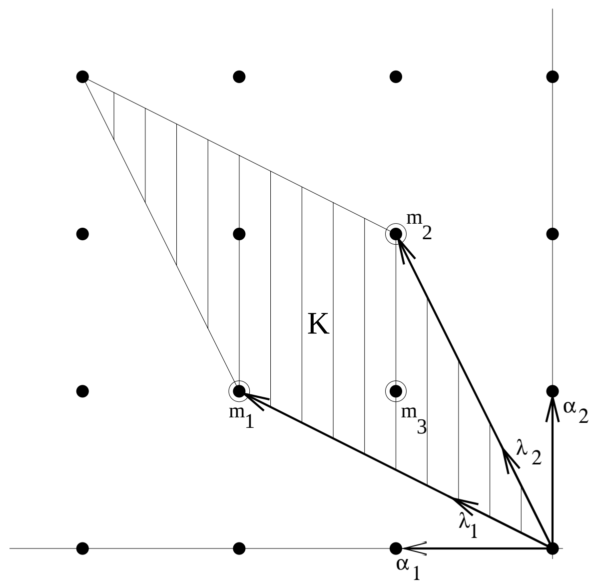
with $\Lambda = \Lambda_{A,G}$ as before.

Finally, we remark that if k is a PID then the Picard group of R is trivial, as is in fact the full projective class group $K_0(R)/\langle [R] \rangle$. This is a consequence of Gubeladze's theorem [7] stating that all projective modules over $R = k[M]$ are free.

2.7. **Examples.** We illustrate the foregoing with a couple of explicit examples. In each case, A will be effective; so $\pi = \mathrm{Id}$ and $M = M_+ = A \cap \Lambda_+$. We will follow the notations in the proof of Theorem (2.4) and in (2.5) quite closely. The input in both examples is a finite group G that is generated by a given collection of integer reflection matrices. The examples may be interpreted as the Weyl groups of the root systems A_2 and A_3 acting on their root lattices, but this information is not needed for the practical calculation of the invariant algebras.

2.7.1. *An example in rank 2.* Let A be free abelian of rank 2, with $\mathbb{Z}$-basis $\{a, b\}$, and let G be the subgroup of $\mathrm{GL}(A) = \mathrm{GL}_2(\mathbb{Z})$ that is generated by the matrices $r = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ and $s = \left(\begin{smallmatrix} 1 & -1 \\ 0 & -1 \end{smallmatrix}\right)$. (These matrices act on the right on A, viewed as integer row vectors of length 2.) The generators r and s are reflections, and $G \cong S_3$, the symmetric group on 3 symbols. The only other reflection in G is $t = \left(\begin{smallmatrix} -1 & 0 \\ -1 & 1 \end{smallmatrix}\right)$; all reflections are conjugate in G, and none are diagonalizable. As a generator for $\mathrm{Ker}_A(g + \mathrm{Id})$, we choose $a_r = (-1, 1) = a^{-1}b$; similarly, we select $a_s = (0, 1) = b$ for s and $a_t = (1, 0) = a$ for t. So $\Phi = \{\pm a_r, \pm a_s, \pm a_t\}$ (a root system of type A_2). As base for Φ, we fix $\Delta = \{\alpha_1 = -a_t = (-1, 0), \alpha_2 = a_s = (0, 1)\}$; so $g_1 = t$ and $g_2 = s$. This leads to the fundamental dominant weights $\lambda_1 = (-2/3, 1/3)$, $\lambda_2 = (-1/3, 2/3)$. The zonotope

$K = [0, m_1] + [0, m_2]$ of (2.5.1) is given by $m_i = 3\lambda_i$, and we obtain the following generators for M: m_1, m_2, and $m_3 = \lambda_1 + \lambda_2$.



Therefore, $\sigma(\lambda_1)^3$, $\sigma(\lambda_2)^3$, and $\sigma(\lambda_1)\sigma(\lambda_2)$, form a fundamental system of invariants in R. Returning to multiplicative notation, the orbit sums for the fundamental dominant weights are:

$$\sigma(\lambda_1) = a^{-2/3}b^{1/3} + a^{1/3}b^{-2/3} + a^{1/3}b^{1/3} = a^{1/3}b^{1/3}(a^{-1} + b^{-1} + 1)$$
$$\sigma(\lambda_2) = a^{-1/3}b^{-1/3} + a^{-1/3}b^{2/3} + a^{2/3}b^{-1/3} = a^{-1/3}b^{-1/3}(a + b + 1)\ .$$

This leads to the following explicit system of fundamental invariants:

$$\mu_1 = \sigma(\lambda_1)^3 = ab(a^{-1} + b^{-1} + 1)^3,$$
$$\mu_2 = \sigma(\lambda_2)^3 = a^{-1}b^{-1}(a + b + 1)^3,$$
$$\mu_3 = \sigma(\lambda_1)\sigma(\lambda_2) = (a + b + 1)(a^{-1} + b^{-1} + 1)\ .$$

The class group of R (over a factorial ring k) evaluates to $\mathrm{Cl}(R) = \Lambda/A \cong \mathbb{Z}/3\mathbb{Z}$.

2.7.2. *Example in rank 3.* Let A be free abelian with $\mathbb{Z}$-basis $\{a, b, c\}$, and let G be the subgroup of $\mathrm{GL}(A) = \mathrm{GL}_3(\mathbb{Z})$ that is generated by the matrices $r = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $s = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & -1 \end{pmatrix}$, and $t = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$. This group is isomorphic to S_4. The generators are reflections; they are all conjugate. The complete set reflections is the full G-conjugacy class: $\{r, s, t, w = r^t, u = s^t, v = s^w\}$; none are diagonalizable. The root system $\Phi = \Phi_{A,G}$ evaluates to $\Phi = \{\pm(1,0,0), \pm(1,0,-1), \pm(1,-1,0), \pm(0,1,0), \pm(0,0,1), \pm(0,1,-1)\}$. A suitable base of Φ is $\Delta = \{\alpha_1 = a_t = (-1,0,1), \alpha_2 = a_r = (1,-1,0), \alpha_3 = a_s = (0,0,-1)\}$;

so $g_1 = t$, $g_2 = r$, $g_3 = s$. This results in the following fundamental dominant weights: $\lambda_1 = (-1/2, -1/2, 1/2)$, $\lambda_2 = (1/4, -3/4, 1/4)$, and $\lambda_3 = (-1/4, -1/4, -1/4)$. The zonotope K is spanned by $m_1 = 2\lambda_1$, $m_2 = 4\lambda_2$, and $m_3 = 4\lambda_3$, and the generators of M are: m_1, m_2, m_3, $m_4 = \lambda_2 + \lambda_3$, $m_5 = \lambda_1 + 2\lambda_2$, and $m_6 = \lambda_1 + 2\lambda_3$. Calculating the orbits sums:

$$\begin{aligned}
\sigma(\lambda_1) &= a^{-1/2}b^{-1/2}c^{-1/2}(a+b+c+ab+ac+bc),\\
\sigma(\lambda_2) &= a^{1/4}b^{1/4}c^{1/4}(a^{-1}+b^{-1}+c^{-1}+1),\\
\sigma(\lambda_3) &= a^{-1/4}b^{-1/4}c^{-1/4}(a+b+c+1).
\end{aligned}$$

This leads to the following explicit system of fundamental invariants:

$$\begin{aligned}
\mu_1 = \sigma(\lambda_1)^2 &= a^{-1}b^{-1}c^{-1}(a+b+c+ab+ac+bc)^2,\\
\mu_2 = \sigma(\lambda_2)^4 &= abc(a^{-1}+b^{-1}+c^{-1}+1)^4,\\
\mu_3 = \sigma(\lambda_3)^4 &= a^{-1}b^{-1}c^{-1}(a+b+c+1)^4,\\
\mu_4 = \sigma(\lambda_2)\sigma(\lambda_3) &= (a+b+c+1)(a^{-1}+b^{-1}+c^{-1}+1),\\
\mu_5 = \sigma(\lambda_1)\sigma(\lambda_2)^2 &= (a+b+c+ab+ac+bc)(a^{-1}+b^{-1}+c^{-1}+1)^2,\\
\mu_6 = \sigma(\lambda_1)\sigma(\lambda_3)^2 &= (a^{-1}+b^{-1}+c^{-1}+a^{-1}b^{-1}+a^{-1}c^{-1}+b^{-1}c^{-1})(a+b+c+1)^2.
\end{aligned}$$

For the class group of R, we obtain $\mathrm{Cl}(R) = \Lambda/A \cong \mathbb{Z}/4\mathbb{Z}$.

The calculations for this example were performed with GAP (version 3.4) [15]; the code is available under `http://www.math.temple.edu/~lorenz/semigroup.html`.

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