

ON THE COHEN-MACAULAY PROPERTY OF MULTIPLICATIVE INVARIANTS

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ABSTRACT. We investigate the Cohen-Macaulay property for rings of invariants under multiplicative actions of a finite group \mathcal{G} . By definition, these are \mathcal{G} -actions on Laurent polynomial algebras $\mathbb{k}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ that stabilize the multiplicative group consisting of all monomials in the variables x_i . For the most part, we concentrate on the case where the base ring \mathbb{k} is \mathbb{Z} . Our main result states that if \mathcal{G} acts non-trivially and the invariant ring $\mathbb{Z}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]^{\mathcal{G}}$ is Cohen-Macaulay then the abelianized isotropy groups $\mathcal{G}_m^{\text{ab}}$ of all monomials m are generated by the bireflections in \mathcal{G}_m and at least one $\mathcal{G}_m^{\text{ab}}$ is non-trivial. As an application, we prove the multiplicative version of Kemper's 3-copies conjecture.

INTRODUCTION

This article is a sequel to [LPk]. Unlike in [LPk], however, our focus will be specifically on multiplicative invariants. In detail, let $L \cong \mathbb{Z}^n$ denote a lattice on which a finite group \mathcal{G} acts by automorphisms. The \mathcal{G} -action on L extends uniquely to an action by \mathbb{k} -algebra automorphisms on the group algebra $\mathbb{k}[L] \cong \mathbb{k}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ over any commutative base ring \mathbb{k} . We are interested in the question when the subalgebra $\mathbb{k}[L]^{\mathcal{G}}$ consisting of all \mathcal{G} -invariant elements of $\mathbb{k}[L]$ has the Cohen-Macaulay property. The reader is assumed to have some familiarity with Cohen-Macaulay rings; a good reference on this subject is [BH].

It is a standard fact that $\mathbb{k}[L]$ is Cohen-Macaulay precisely if \mathbb{k} is. On the other hand, while $\mathbb{k}[L]^{\mathcal{G}}$ can only be Cohen-Macaulay when \mathbb{k} is so, the latter condition is far from sufficient and rather stringent additional conditions on the action of \mathcal{G} on L are required to ensure that $\mathbb{k}[L]^{\mathcal{G}}$ is Cohen-Macaulay. Remarkably, the question whether or not $\mathbb{k}[L]^{\mathcal{G}}$ is Cohen-Macaulay, for any given base ring \mathbb{k} , depends only on the rational isomorphism class

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of the lattice L , that is, the isomorphism class of $L \otimes_{\mathbb{Z}} \mathbb{Q}$ as $\mathbb{Q}[\mathcal{G}]$ -module; see Proposition 3.4 below. This is in striking contrast with most other ring theoretic properties of $\mathbb{k}[L]^{\mathcal{G}}$ (e.g., regularity, structure of the class group) which tend to be sensitive to the \mathbb{Z} -type of L . For an overview, see [L₁].

We will largely concentrate on the case where the base ring \mathbb{k} is \mathbb{Z} . This is justified in part by the fact that if $\mathbb{Z}[L]^{\mathcal{G}}$ is Cohen-Macaulay then likewise is $\mathbb{k}[L]^{\mathcal{G}}$ for any Cohen-Macaulay base ring \mathbb{k} (Lemma 3.2). Assuming $\mathbb{Z}[L]^{\mathcal{G}}$ to be Cohen-Macaulay, we aim to derive group theoretical consequences for the isotropy groups $\mathcal{G}_m = \{g \in \mathcal{G} \mid g(m) = m\}$ with $m \in L$. An element $g \in \mathcal{G}$ will be called a k -reflection on L if the sublattice $[g, L] = \{g(m) - m \mid m \in L\}$ of L has rank at most k or, equivalently, if the g -fixed points of the \mathbb{Q} -space $L \otimes_{\mathbb{Z}} \mathbb{Q}$ have codimension at most k . As usual, k -reflections with $k = 1$ and $k = 2$ will be referred to as *reflections* and *bireflections*. For any subgroup $\mathcal{H} \leq \mathcal{G}$, we let $\mathcal{H}^{(2)}$ denote the subgroup generated by the elements of \mathcal{H} that act as bireflections on L . Our main result now reads as follows.

Theorem. *Assume that $\mathbb{Z}[L]^{\mathcal{G}}$ is Cohen-Macaulay. Then $\mathcal{G}_m/\mathcal{G}_m^{(2)}$ is a perfect group (i.e., equal to its commutator subgroup) for all $m \in L$. If \mathcal{G} acts non-trivially on L then some \mathcal{G}_m is non-perfect.*

It would be interesting to determine if the conclusion of the theorem can be strengthened to the effect that all isotropy groups \mathcal{G}_m are in fact generated by bireflections on L . I do not know if, for the latter to occur, it is sufficient that \mathcal{G} is generated by bireflections. The corresponding fact for reflection groups is known to be true: if \mathcal{G} is generated by reflections on L (or, equivalently, on $L \otimes_{\mathbb{Z}} \mathbb{Q}$) then so are all isotropy groups \mathcal{G}_m ; see [St, Theorem 1.5] or [Bou₁, Exercise 8(a) on p. 139].

There is essentially a complete classification of finite linear groups generated by bireflections. In arbitrary characteristic, this is due to Guralnick and Saxl [GuS]; for the case of characteristic zero, see Huffman and Wales [HuW]. Bireflection groups have been of interest in connection with the problem of determining all finite linear groups whose algebra of polynomial invariants is a complete intersection. Specifically, suppose that $\mathcal{G} \leq \mathrm{GL}(V)$ for some finite-dimensional vector space V and let $\mathcal{O}(V) = \mathbb{S}(V^*)$ denote the algebra of polynomial functions on V . It was shown by Kac and Watanabe [KW] and independently by Gordeev [G₁] that if the algebra $\mathcal{O}(V)^{\mathcal{G}}$ of all \mathcal{G} -invariant polynomial functions is a complete intersection then \mathcal{G} is generated by bireflections on V . The classification of all groups \mathcal{G} so that $\mathcal{O}(V)^{\mathcal{G}}$ is a complete intersection has been achieved by Gordeev [G₂] and by Nakajima [N].

The last assertion of the above Theorem implies in particular that if $\mathbb{Z}[L]^{\mathcal{G}}$ is Cohen-Macaulay and \mathcal{G} acts non-trivially on L then some element of

\mathcal{G} acts as a non-trivial bireflection on L . Hence we obtain the following multiplicative version of Kemper’s 3-copies conjecture:

Corollary. *If \mathcal{G} acts non-trivially on L and $r \geq 3$ then $\mathbb{Z}[L^{\oplus r}]^{\mathcal{G}}$ is not Cohen-Macaulay.*

The 3-copies conjecture was formulated by Kemper [K₁, Vermutung 3.12] in the context of polynomial invariants. Using the above notation, the original conjecture states that if $1 \neq \mathcal{G} \leq \mathrm{GL}(V)$ and the characteristic of the base field of V divides the order of \mathcal{G} (“modular case”) then the invariant algebra $\mathcal{O}(V^{\oplus r})^{\mathcal{G}}$ will not be Cohen-Macaulay for any $r \geq 3$. This is still open. The main factors contributing to our success in the multiplicative case are the following:

- Multiplicative actions are permutation actions: \mathcal{G} permutes the \mathbb{k} -basis of $\mathbb{k}[L]$ consisting of all “monomials”, corresponding to the elements of the lattice L . Consequently, the cohomology $H^*(\mathcal{G}, \mathbb{k}[L])$ is simply the direct sum of the various $H^*(\mathcal{G}_m, \mathbb{k})$ with m running over a transversal for the \mathcal{G} -orbits in L .
- Up to conjugacy, there are only finitely many finite subgroups of $\mathrm{GL}_n(\mathbb{Z})$ and these groups are explicitly known for small n . A crucial observation for our purposes is the following: if \mathcal{G} is a nontrivial finite perfect subgroup of $\mathrm{GL}_n(\mathbb{Z})$ such that no $1 \neq g \in \mathcal{G}$ has eigenvalue 1 then \mathcal{G} is isomorphic to the binary icosahedral group and $n \geq 8$; see Lemma 2.3 below.

A brief outline of the contents of this article is as follows. The short preliminary Section 1 is devoted to general actions of a finite group \mathcal{G} on a commutative ring R . This material relies rather heavily on [LPk]. We liberate a technical result from [LPk] from any a priori hypotheses on the characteristic; the new version (Proposition 1.4) states that if R and $R^{\mathcal{G}}$ are both Cohen-Macaulay and $H^i(\mathcal{G}, R) = 0$ for $0 < i < k$ then $H^k(\mathcal{G}, R)$ is detected by $k + 1$ -reflections. Section 2 then specializes to the case of multiplicative actions. We assemble the main tools required for the proof of the Theorem, which is presented in Section 3. The article concludes with a brief discussion of possible avenues for further investigation and some examples.

1. FINITE GROUP ACTIONS ON RINGS

1.1. In this section, R will be a commutative ring on which a finite group \mathcal{G} acts by ring automorphisms $r \mapsto g(r)$ ($r \in R, g \in \mathcal{G}$). The subring of \mathcal{G} -invariant elements of R will be denoted by $R^{\mathcal{G}}$.

1.2. Generalized reflections. Following [GK], we will say an element $g \in \mathcal{G}$ acts as a k -reflection on R if g belongs to the inertia group

$$I_{\mathcal{G}}(\mathfrak{P}) = \{g \in \mathcal{G} \mid g(r) - r \in \mathfrak{P} \ \forall r \in R\}$$

of some prime ideal $\mathfrak{P} \in \text{Spec } R$ with $\text{height } \mathfrak{P} \leq k$. The cases $k = 1$ and $k = 2$ will be referred to as *reflections* and *bireflections*, respectively. Define the ideal $I_R(g)$ of R by

$$I_R(g) = \sum_{r \in R} (g(r) - r)R.$$

Evidently, $\mathfrak{P} \supseteq I_R(g)$ is equivalent to $g \in I_{\mathcal{G}}(\mathfrak{P})$. Thus:

$$g \text{ is a } k\text{-reflection on } R \text{ if and only if } \text{height } I_R(g) \leq k.$$

For each subgroup $\mathcal{H} \leq \mathcal{G}$, we put

$$I_R(\mathcal{H}) = \sum_{g \in \mathcal{H}} I_R(g).$$

It suffices to let g run over a set of generators of the group \mathcal{H} in this sum.

1.3. A height estimate. The cohomology $H^*(\mathcal{G}, R) = \bigoplus_{n \geq 0} H^n(\mathcal{G}, R)$ has a canonical $R^{\mathcal{G}}$ -module structure: for each $r \in R^{\mathcal{G}}$, the map $\rho: R \rightarrow R$, $s \mapsto rs$, is \mathcal{G} -equivariant and hence it induces a map on cohomology $\rho_*: H^*(\mathcal{G}, R) \rightarrow H^*(\mathcal{G}, R)$. The element r acts on $H^*(\mathcal{G}, R)$ via ρ_* . Let $\text{res}_{\mathcal{H}}^{\mathcal{G}}: H^*(\mathcal{G}, R) \rightarrow H^*(\mathcal{H}, R)$ denote the restriction map.

The following lemma extends [LPk, Proposition 1.4].

Lemma. *For any $x \in H^*(\mathcal{G}, R)$,*

$$\text{height } \text{ann}_{R^{\mathcal{G}}}(x) \geq \inf\{\text{height } I_R(\mathcal{H}) \mid \mathcal{H} \leq \mathcal{G}, \text{res}_{\mathcal{H}}^{\mathcal{G}}(x) \neq 0\}.$$

Proof. Put $\mathfrak{X} = \{\mathcal{H} \leq \mathcal{G} \mid \text{res}_{\mathcal{H}}^{\mathcal{G}}(x) = 0\}$. For each $\mathcal{H} \leq \mathcal{G}$, let $R_{\mathcal{H}}^{\mathcal{G}}$ denote the image of the relative trace map $R^{\mathcal{H}} \rightarrow R^{\mathcal{G}}$, $r \mapsto \sum_g g(r)$, where g runs over a transversal for the cosets $g\mathcal{H}$ of \mathcal{H} in \mathcal{G} . By [LPk, Lemma 1.3],

$$R_{\mathcal{H}}^{\mathcal{G}} \subseteq \text{ann}_{R^{\mathcal{G}}}(x) \quad \text{for all } \mathcal{H} \in \mathfrak{X}.$$

To prove the lemma, we may assume that $\text{ann}_{R^{\mathcal{G}}}(x)$ is a proper ideal of $R^{\mathcal{G}}$; for, otherwise $\text{height } \text{ann}_{R^{\mathcal{G}}}(x) = \infty$. Choose a prime ideal \mathfrak{p} of $R^{\mathcal{G}}$ with $\mathfrak{p} \supseteq \text{ann}_{R^{\mathcal{G}}}(x)$ and $\text{height } \mathfrak{p} = \text{height } \text{ann}_{R^{\mathcal{G}}}(x)$. If \mathfrak{P} is a prime of R that lies over \mathfrak{p} then

$$R_{\mathcal{H}}^{\mathcal{G}} \subseteq \mathfrak{P} \quad \text{for all } \mathcal{H} \in \mathfrak{X}$$

and $\text{height } \mathfrak{P} = \text{height } \mathfrak{p}$. By [LPk, Lemma 1.1], the above inclusion implies that

$$[I_{\mathcal{G}}(\mathfrak{P}) : I_{\mathcal{H}}(\mathfrak{P})] \in \mathfrak{P} \quad \text{for all } \mathcal{H} \in \mathfrak{X}.$$

Put $p = \text{char } R/\mathfrak{P}$ and let $\mathcal{P} \leq I_{\mathcal{G}}(\mathfrak{P})$ be a Sylow p -subgroup of $I_{\mathcal{G}}(\mathfrak{P})$ (so $\mathcal{P} = 1$ if $p = 0$). Then $I_R(\mathcal{P}) \subseteq \mathfrak{P}$ and $[I_{\mathcal{G}}(\mathfrak{P}) : \mathcal{P}] \notin \mathfrak{P}$. Hence,

$\mathcal{P} \notin \mathfrak{X}$ and $\text{height } I_R(\mathcal{P}) \leq \text{height } \mathfrak{P} = \text{height } \text{ann}_{R^{\mathcal{G}}}(x)$. This proves the lemma. \square

We remark that the lemma and its proof carry over verbatim to the more general situation where $H^*(\mathcal{G}, R)$ is replaced by $H^*(\mathcal{G}, M)$, where M is some module over the skew group ring of \mathcal{G} over R ; cf. [LPk]. However, we will not be concerned with this generalization here.

1.4. A necessary condition. In this section, we assume that R is noetherian as $R^{\mathcal{G}}$ -module. This assumption is satisfied whenever R is an affine algebra over some noetherian subring $\mathbb{k} \subseteq R^{\mathcal{G}}$; see [Bou₂, Théorème 2 on p. 33]. Put

$$\mathfrak{X}_k = \{\mathcal{H} \leq \mathcal{G} \mid \text{height } I_R(\mathcal{H}) \leq k\}. \quad (1.1)$$

Note that each $\mathcal{H} \in \mathfrak{X}_k$ consists of k -reflections on R . The following proposition is a characteristic-free version of [LPk, Proposition 4.1].

Proposition. *Assume that R and $R^{\mathcal{G}}$ are Cohen-Macaulay. If $H^i(\mathcal{G}, R) = 0$ ($0 < i < k$) then the restriction map*

$$\text{res}_{\mathfrak{X}_{k+1}}^{\mathcal{G}} : H^k(\mathcal{G}, R) \rightarrow \prod_{\mathcal{H} \in \mathfrak{X}_{k+1}} H^k(\mathcal{H}, R)$$

is injective.

Proof. We may assume that $H^k(\mathcal{G}, R) \neq 0$. Let $x \in H^k(\mathcal{G}, R)$ be nonzero and put $\mathfrak{a} = \text{ann}_{R^{\mathcal{G}}}(x)$. By [LPk, Proposition 3.3], $\text{depth } \mathfrak{a} \leq k + 1$. Since $R^{\mathcal{G}}$ is Cohen-Macaulay, $\text{depth } \mathfrak{a} = \text{height } \mathfrak{a}$. Thus, Lemma 1.3 implies that $k + 1 \geq \text{height } I_R(\mathcal{H})$ for some $\mathcal{H} \leq \mathcal{G}$ with $\text{res}_{\mathcal{H}}^{\mathcal{G}}(x) \neq 0$. The proposition follows. \square

Note that the vanishing hypothesis on $H^i(\mathcal{G}, R)$ is vacuous for $k = 1$. Thus, $H^1(\mathcal{G}, R)$ is detected by bireflections whenever R and $R^{\mathcal{G}}$ are both Cohen-Macaulay.

2. MULTIPLICATIVE ACTIONS

2.1. For the remainder of this article, L will denote a lattice on which the finite group \mathcal{G} acts by automorphisms $m \mapsto g(m)$ ($m \in L, g \in \mathcal{G}$). The group algebra of L over some commutative base ring \mathbb{k} will be denoted by $\mathbb{k}[L]$. We will use additive notation in L . The \mathbb{k} -basis element of $\mathbb{k}[L]$ corresponding to the lattice element $m \in L$ will be written as

$$\mathbf{x}^m ;$$

so $\mathbf{x}^0 = 1$, $\mathbf{x}^{m+m'} = \mathbf{x}^m \mathbf{x}^{m'}$, and $\mathbf{x}^{-m} = (\mathbf{x}^m)^{-1}$. The action of \mathcal{G} on L extends uniquely to an action by \mathbb{k} -algebra automorphisms on $\mathbb{k}[L]$:

$$g\left(\sum_{m \in L} k_m \mathbf{x}^m\right) = \sum_{m \in L} k_m \mathbf{x}^{g(m)}.$$

The invariant algebra $\mathbb{k}[L]^{\mathcal{G}}$ is a free \mathbb{k} -module: a \mathbb{k} -basis is given by the \mathcal{G} -orbit sums $\sigma(m) = \sum_{m' \in \mathcal{G}(m)} \mathbf{x}^{m'}$, where $\mathcal{G}(m)$ denotes the \mathcal{G} -orbit of $m \in L$. Since all orbit sums are defined over \mathbb{Z} , we have

$$\mathbb{k}[L]^{\mathcal{G}} = \mathbb{k} \otimes_{\mathbb{Z}} \mathbb{Z}[L]^{\mathcal{G}}. \quad (2.1)$$

2.2. Let \mathcal{H} be a subgroup of \mathcal{G} . We compute the height of the ideal $I_{\mathbb{k}[L]}(\mathcal{H})$ from §1.2. Let

$$L^{\mathcal{H}} = \{m \in L \mid g(m) = m \text{ for all } g \in \mathcal{H}\}$$

denote the lattice of \mathcal{H} -invariants in L and define the sublattice $[\mathcal{H}, L]$ of L by

$$[\mathcal{H}, L] = \sum_{g \in \mathcal{H}} [g, L],$$

where $[g, L] = \{g(m) - m \mid m \in L\}$. It suffices to let g run over a set of generators of the group \mathcal{H} in the above formulas.

Lemma. *With the above notation, $\mathbb{k}[L]/I_{\mathbb{k}[L]}(\mathcal{H}) \cong \mathbb{k}[L]/[\mathcal{H}, L]$ and*

$$\text{height } I_{\mathbb{k}[L]}(\mathcal{H}) = \text{rank}[\mathcal{H}, L] = \text{rank } L - \text{rank } L^{\mathcal{H}}.$$

Proof. Since the ideal $I_{\mathbb{k}[L]}(\mathcal{H})$ is generated by the elements $\mathbf{x}^{g(m)-m} - 1$ with $m \in L$ and $g \in \mathcal{H}$, the isomorphism $\mathbb{k}[L]/I_{\mathbb{k}[L]}(\mathcal{H}) \cong \mathbb{k}[L]/[\mathcal{H}, L]$ is clear.

To prove the equality $\text{rank}[\mathcal{H}, L] = \text{rank } L - \text{rank } L^{\mathcal{H}}$, note that the rational group algebra $\mathbb{Q}[\mathcal{H}]$ is the direct sum of the ideals $\mathbb{Q}\left(\sum_{g \in \mathcal{H}} g\right)$ and $\sum_{g \in \mathcal{H}} \mathbb{Q}(g - 1)$. This implies $L \otimes_{\mathbb{Z}} \mathbb{Q} = (L^{\mathcal{H}} \otimes_{\mathbb{Z}} \mathbb{Q}) \oplus ([\mathcal{H}, L] \otimes_{\mathbb{Z}} \mathbb{Q})$. Hence, $\text{rank } L = \text{rank } L^{\mathcal{H}} + \text{rank}[\mathcal{H}, L]$.

To complete the proof, it suffices to show that

$$\text{height } \mathfrak{P} = \text{rank}[\mathcal{H}, L]$$

holds for any minimal covering prime \mathfrak{P} of $I_{\mathbb{k}[L]}(\mathcal{H})$. Put $A = L/[\mathcal{H}, L]$ and $\overline{\mathfrak{P}} = \mathfrak{P}/I_{\mathbb{k}[L]}(\mathcal{H})$, a minimal prime of $\mathbb{k}[L]/I_{\mathbb{k}[L]}(\mathcal{H}) = \mathbb{k}[A]$. Further, put $\mathfrak{p} = \overline{\mathfrak{P}} \cap \mathbb{k} = \mathfrak{P} \cap \mathbb{k}$. Since the extension $\mathbb{k} \hookrightarrow \mathbb{k}[A] = \mathbb{k}[L]/I_{\mathbb{k}[L]}(\mathcal{H})$ is free, \mathfrak{p} is a minimal prime of \mathbb{k} ; see [Bou₃, Cor. on p. AC VIII.15]. Hence, descending chains of primes in $\mathbb{k}[L]$ starting with \mathfrak{P} correspond in a 1-to-1 fashion to descending chains of primes of $Q(\mathbb{k}/\mathfrak{p})[L]$ starting with the prime

that is generated by \mathfrak{P} . Thus, replacing \mathbb{k} by $Q(\mathbb{k}/\mathfrak{p})$, we may assume that \mathbb{k} is a field. But then

$$\text{height } \mathfrak{P} = \dim \mathbb{k}[L] - \dim \mathbb{k}[L]/\mathfrak{P} = \text{rank } L - \dim \mathbb{k}[L]/\mathfrak{P}.$$

Let $\overline{\mathfrak{P}}_0 = \overline{\mathfrak{P}} \cap \mathbb{k}[A_0]$, where A_0 denotes the torsion subgroup of A . Since $\overline{\mathfrak{P}}$ is minimal, we have $\overline{\mathfrak{P}} = \overline{\mathfrak{P}}_0 \mathbb{k}[A]$ and so $\mathbb{k}[L]/\mathfrak{P} \cong \mathbb{k}_0[A/A_0]$, where $\mathbb{k}_0 = \mathbb{k}[A_0]/\overline{\mathfrak{P}}_0$ is a field. Thus, $\dim \mathbb{k}[L]/\mathfrak{P} = \text{rank } A/A_0$. Finally, $\text{rank } A/A_0 = \text{rank } A = \text{rank } L - \text{rank}[\mathcal{H}, L]$, which completes the proof. \square

Specializing the lemma to the case where $\mathcal{H} = \langle g \rangle$ for some $g \in \mathcal{G}$, we see that g acts as a k -reflection on $\mathbb{k}[L]$ if and only if g acts as a k -reflection on L , that is,

$$\text{rank}[g, L] \leq k.$$

Moreover, the collection of subgroups \mathfrak{X}_k in equation (1.1) can now be written as

$$\mathfrak{X}_k = \{\mathcal{H} \leq \mathcal{G} \mid \text{rank } L/L^{\mathcal{H}} \leq k\}. \quad (2.2)$$

2.3. Fixed-point-free lattices for perfect groups. The \mathcal{G} -action on L is called *fixed-point-free* if $g(m) \neq m$ holds for all $0 \neq m \in L$ and $1 \neq g \in \mathcal{G}$. Recall also that the group \mathcal{G} is said to be *perfect* if $\mathcal{G}^{\text{ab}} = \mathcal{G}/[\mathcal{G}, \mathcal{G}] = 1$.

Lemma. *Assume that \mathcal{G} is a nontrivial perfect group acting fixed-point-freely on the nonzero lattice L . Then \mathcal{G} is isomorphic to the binary icosahedral group $2.\mathcal{A}_5 \cong \text{SL}_2(\mathbb{F}_5)$ and $\text{rank } L$ is a multiple of 8.*

Proof. Put $V = L \otimes_{\mathbb{Z}} \mathbb{C}$, a nonzero fixed-point-free $\mathbb{C}[\mathcal{G}]$ -module. By a well-known theorem of Zassenhaus (see [Wo, Theorem 6.2.1]), \mathcal{G} is isomorphic to the binary icosahedral group $2.\mathcal{A}_5$ and the irreducible constituents of V are 2-dimensional. The binary icosahedral group has two irreducible complex representations of degree 2; they are Galois conjugates of each other and both have Frobenius-Schur indicator -1 . We denote the corresponding $\mathbb{C}[\mathcal{G}]$ -modules by V_1 and V_2 . Both V_i occur with the same multiplicity in V , since V is defined over \mathbb{Q} . Thus, $V \cong (V_1 \oplus V_2)^m$ for some m and $\text{rank } L = 4m$. We have to show that m is even. Since both V_i have indicator -1 , it follows that $V_1 \oplus V_2$ is not defined over \mathbb{R} , whereas each V_i^2 is defined over \mathbb{R} ; see [I, (9.21)]. Thus, $V_1 \oplus V_2$ represents an element x of order 2 in the cokernel of the scalar extension map $G_0(\mathbb{R}[\mathcal{G}]) \rightarrow G_0(\mathbb{C}[\mathcal{G}])$, and $mx = 0$. Therefore, m must be even, as desired. \square

We remark that the binary icosahedral group $2.\mathcal{A}_5$ is isomorphic to the subgroup of the nonzero quaternions \mathbb{H}^* that is generated by $(a+i+ja^*)/2$ and $(a+j+ka^*)/2$, where $a = (1 + \sqrt{5})/2$ and $a^* = (1 - \sqrt{5})/2$ and $\{1, i, j, k\}$ is the standard \mathbb{R} -basis of \mathbb{H} . Thus, letting $2.\mathcal{A}_5$ act on \mathbb{H} via left

multiplication, \mathbb{H} becomes a 2-dimensional fixed-point-free complex representation of $2.\mathcal{A}_5$. It is easy to see that this representation can be realized over $K = \mathbb{Q}(i, \sqrt{5})$; so $\mathbb{H} = V \otimes_K \mathbb{C}$ with $\dim_{\mathbb{Q}} V = 2[K : \mathbb{Q}] = 8$. Any $2.\mathcal{A}_5$ -lattice for V will be fixed-point-free and have rank 8.

2.4. Isotropy groups. The isotropy group of an element $m \in L$ in \mathcal{G} will be denoted by \mathcal{G}_m ; so

$$\mathcal{G}_m = \{g \in \mathcal{G} \mid g(m) = m\}.$$

The \mathcal{G} -lattice L is called *faithful* if $\text{Ker}_{\mathcal{G}}(L) = \bigcap_{m \in L} \mathcal{G}_m = 1$. The following lemma, at least part (a), is well-known. We include the proof for the reader's convenience.

- Lemma.** (a) *The set of isotropy groups $\{\mathcal{G}_m \mid m \in L\}$ is closed under conjugation and under taking intersections.*
 (b) *Assume that the \mathcal{G} -lattice L is faithful. If \mathcal{G}_m ($m \in L$) is a minimal non-identity isotropy group then \mathcal{G}_m acts fixed-point-freely on $L/L^{\mathcal{G}_m} \neq 0$.*

Proof. Consider the $\mathbb{Q}[\mathcal{G}]$ -module $V = L \otimes_{\mathbb{Z}} \mathbb{Q}$. The collection of isotropy groups \mathcal{G}_m remains unchanged when allowing $m \in V$. Moreover, for any subgroup $\mathcal{H} \leq \mathcal{G}$, $L/L^{\mathcal{H}}$ is an \mathcal{H} -lattice with $L/L^{\mathcal{H}} \otimes_{\mathbb{Z}} \mathbb{Q} \cong V/V^{\mathcal{H}}$.

(a) The first assertion is clear, since ${}^g\mathcal{G}_m = \mathcal{G}_{g(m)}$ holds for all $g \in \mathcal{G}, m \in V$. For the second assertion, let M be a non-empty subset of V and put $\mathcal{G}_M = \bigcap_{m \in M} \mathcal{G}_m$. We must show that $\mathcal{G}_M = \mathcal{G}_m$ for some $m \in V$. Put $W = V^{\mathcal{G}_M}$. If $g \in \mathcal{G} \setminus \mathcal{G}_M$ then $W^g = \{w \in W \mid g(w) = w\}$ is a proper subspace of W , since some element of M does not belong to W^g . Any $m \in W \setminus \bigcup_{g \in \mathcal{G} \setminus \mathcal{G}_M} W^g$ satisfies $\mathcal{G}_m = \mathcal{G}_M$.

(b) Let $\mathcal{H} = \mathcal{G}_m$ be a minimal non-identity member of $\{\mathcal{G}_m \mid m \in V\}$. As $\mathbb{Q}[\mathcal{H}]$ -modules, we may identify V and $V^{\mathcal{H}} \oplus V/V^{\mathcal{H}}$. If $0 \neq v \in V/V^{\mathcal{H}}$ then $\mathcal{H}_v = \mathcal{H} \cap \mathcal{G}_v \subsetneq \mathcal{H}$. In view of (a), our minimality assumption on \mathcal{H} forces $\mathcal{H}_v = 1$. Thus, \mathcal{H} acts fixed-point-freely on $V/V^{\mathcal{H}}$, and hence on $L/L^{\mathcal{H}}$. \square

Proposition. *Assume that L is a faithful \mathcal{G} -lattice such that all minimal isotropy groups $1 \neq \mathcal{G}_m$ ($m \in L$) are perfect. Then $\text{rank } L/L^{\mathcal{H}} \geq 8$ holds for every nonidentity subgroup $\mathcal{H} \leq \mathcal{G}$.*

In the notation of equation (2.2), the conclusion of the proposition can be stated as follows:

$$\mathfrak{X}_k = \{1\} \text{ for all } k < 8.$$

Proof of the Proposition. Put $\overline{\mathcal{H}} = \bigcap_{m \in L^{\mathcal{H}}} \mathcal{G}_m$. Then $\overline{\mathcal{H}} \supseteq \mathcal{H}$ and $L^{\overline{\mathcal{H}}} = L^{\mathcal{H}}$. Lemma 2.4(a) further implies that $\overline{\mathcal{H}} = \mathcal{G}_m$ for some m . Replacing \mathcal{H} by $\overline{\mathcal{H}}$, we may assume that \mathcal{H} is a nonidentity isotropy group. If

\mathcal{H} is not minimal then replace \mathcal{H} by a smaller nonidentity isotropy group; this does not increase the value of $\text{rank } L/L^{\mathcal{H}}$. Thus, we may assume that \mathcal{H} is a minimal nonidentity isotropy group, and hence \mathcal{H} is perfect. By Lemma 2.4(b), \mathcal{H} acts fixed-point-freely on $L/L^{\mathcal{H}} \neq 0$ and Lemma 2.3 implies that $\text{rank } L/L^{\mathcal{H}} \geq 8$, proving the proposition. \square

2.5. Cohomology. Let \mathfrak{X} denote any collection of subgroups of \mathcal{G} that is closed under conjugation and under taking subgroups. We will investigate injectivity of the restriction map

$$\text{res}_{\mathfrak{X}}^{\mathcal{G}}: H^k(\mathcal{G}, \mathbb{k}[L]) \rightarrow \prod_{\mathcal{H} \in \mathfrak{X}} H^k(\mathcal{H}, \mathbb{k}[L]).$$

This map was considered in Proposition 1.4 for $\mathfrak{X} = \mathfrak{X}_{k+1}$.

Lemma. *The map $\text{res}_{\mathfrak{X}}^{\mathcal{G}}: H^k(\mathcal{G}, \mathbb{k}[L]) \rightarrow \prod_{\mathcal{H} \in \mathfrak{X}} H^k(\mathcal{H}, \mathbb{k}[L])$ is injective if and only if the restriction maps*

$$H^k(\mathcal{G}_m, \mathbb{k}) \rightarrow \prod_{\substack{\mathcal{H} \in \mathfrak{X} \\ \mathcal{H} \leq \mathcal{G}_m}} H^k(\mathcal{H}, \mathbb{k})$$

are injective for all $m \in L$.

Proof. As $\mathbb{k}[\mathcal{G}]$ -module, $\mathbb{k}[L]$ is a permutation module:

$$\mathbb{k}[L] \cong \bigoplus_{m \in \mathcal{G} \backslash L} \mathbb{k}[\mathcal{G}/\mathcal{G}_m],$$

where $\mathbb{k}[\mathcal{G}/\mathcal{G}_m] = \mathbb{k}[\mathcal{G}] \otimes_{\mathbb{k}[\mathcal{G}_m]} \mathbb{k}$ and $\mathcal{G} \backslash L$ is a transversal for the \mathcal{G} -orbits in L . For each subgroup $\mathcal{H} \leq \mathcal{G}$,

$$\mathbb{k}[\mathcal{G}/\mathcal{G}_m]_{|\mathcal{H}} \cong \bigoplus_{g \in \mathcal{H} \backslash \mathcal{G}/\mathcal{G}_m} \mathbb{k}[\mathcal{H}/{}^g\mathcal{G}_m \cap \mathcal{H}];$$

see [CR, 10.13]. Therefore, $\text{res}_{\mathcal{H}}^{\mathcal{G}}$ is the direct sum of the restriction maps

$$H^k(\mathcal{G}, \mathbb{k}[\mathcal{G}/\mathcal{G}_m]) \rightarrow H^k(\mathcal{H}, \mathbb{k}[\mathcal{G}/\mathcal{G}_m]) = \bigoplus_{g \in \mathcal{H} \backslash \mathcal{G}/\mathcal{G}_m} H^k(\mathcal{H}, \mathbb{k}[\mathcal{H}/{}^g\mathcal{G}_m \cap \mathcal{H}]).$$

By the Eckmann-Shapiro Lemma [Br, III(5.2),(6.2)], $H^k(\mathcal{G}, \mathbb{k}[\mathcal{G}/\mathcal{G}_m]) \cong H^k(\mathcal{G}_m, \mathbb{k})$ and $H^k(\mathcal{H}, \mathbb{k}[\mathcal{H}/{}^g\mathcal{G}_m \cap \mathcal{H}]) \cong H^k({}^g\mathcal{G}_m \cap \mathcal{H}, \mathbb{k})$. In terms of these isomorphisms, the above restriction map becomes

$$\begin{aligned} \rho_{\mathcal{H},m}: H^k(\mathcal{G}_m, \mathbb{k}) &\rightarrow \bigoplus_{g \in \mathcal{H} \backslash \mathcal{G}/\mathcal{G}_m} H^k({}^g\mathcal{G}_m \cap \mathcal{H}, \mathbb{k}) \\ [f] &\mapsto ([\underline{h} \mapsto f(g^{-1}\underline{h}g)])_g \end{aligned}$$

where $[\cdot]$ denotes the cohomology class of a k -cocycle and \underline{h} stands for a k -tuple of elements of ${}^g\mathcal{G}_m \cap \mathcal{H}$. Therefore,

$$\text{Ker } \rho_{\mathcal{H},m} = \bigcap_{g \in \mathcal{H} \setminus \mathcal{G}/\mathcal{G}_m} \text{Ker} \left(\text{res}_{\mathcal{G}_m \cap \mathcal{H}^g}^{\mathcal{G}_m} : H^k(\mathcal{G}_m, \mathbb{k}) \rightarrow H^k(\mathcal{G}_m \cap \mathcal{H}^g, \mathbb{k}) \right) .$$

Thus, $\text{Ker } \text{res}_{\mathfrak{X}}^{\mathcal{G}}$ is isomorphic to the direct sum of the kernels of the restriction maps

$$H^k(\mathcal{G}_m, \mathbb{k}) \rightarrow \prod_{\mathcal{H} \in \mathfrak{X}} H^k(\mathcal{G}_m \cap \mathcal{H}^g, \mathbb{k})$$

with $m \in \mathcal{G} \setminus L$. Finally, by hypothesis on \mathfrak{X} , the groups $\mathcal{G}_m \cap \mathcal{H}^g$ with $\mathcal{H} \in \mathfrak{X}$ are exactly the groups $\mathcal{H} \in \mathfrak{X}$ with $\mathcal{H} \leq \mathcal{G}_m$. The lemma follows. \square

Corollary. *Let $\mathbb{k} = \mathbb{Z}/(|\mathcal{G}|)$ and $k = 1$. Then $\text{res}_{\mathfrak{X}}^{\mathcal{G}}$ is injective if and only if all $\mathcal{G}_m^{\text{ab}}$ ($m \in L$) are generated by the images of the subgroups $\mathcal{H} \leq \mathcal{G}_m$ with $\mathcal{H} \in \mathfrak{X}$.*

Proof. By the lemma with $k = 1$, the hypothesis on the restriction map says that all restrictions

$$H^1(\mathcal{G}_m, \mathbb{k}) \rightarrow \prod_{\substack{\mathcal{H} \in \mathfrak{X} \\ \mathcal{H} \leq \mathcal{G}_m}} H^1(\mathcal{H}, \mathbb{k})$$

are injective. Now, for each $\mathcal{H} \leq \mathcal{G}$, $H^1(\mathcal{H}, \mathbb{k}) = \text{Hom}(\mathcal{H}^{\text{ab}}, \mathbb{k}) \cong \mathcal{H}^{\text{ab}}$, where the last isomorphism holds by our choice of \mathbb{k} . Therefore, injectivity of the above map is equivalent to $\mathcal{G}_m^{\text{ab}}$ being generated by the images of all $\mathcal{H} \leq \mathcal{G}_m$ with $\mathcal{H} \in \mathfrak{X}$. \square

3. THE COHEN-MACAULAY PROPERTY

3.1. Continuing with the notation of §2.1, we now turn to the question when the invariant algebra $\mathbb{k}[L]^{\mathcal{G}}$ is Cohen-Macaulay. Our principal tool will be Proposition 1.4. We remark that the Cohen-Macaulay hypothesis of Proposition 1.4 simplifies slightly in the setting of multiplicative actions: it suffices to assume that $\mathbb{k}[L]^{\mathcal{G}}$ is Cohen-Macaulay. Indeed, in this case the base ring \mathbb{k} is also Cohen-Macaulay, because $\mathbb{k}[L]^{\mathcal{G}}$ is free over \mathbb{k} , and then $\mathbb{k}[L]$ is Cohen-Macaulay as well; see [BH, Exercise 2.1.23 and Theorems 2.1.9, 2.1.3(b)].

3.2. **Base rings.** Our main interest is in the case where $\mathbb{k} = \mathbb{Z}$. As the following lemma shows, if $\mathbb{Z}[L]^{\mathcal{G}}$ is Cohen-Macaulay then so is $\mathbb{k}[L]^{\mathcal{G}}$ for any Cohen-Macaulay base ring \mathbb{k} .

Lemma. *The following are equivalent:*

- (a) $\mathbb{Z}[L]^{\mathcal{G}}$ is Cohen-Macaulay;
- (b) $\mathbb{k}[L]^{\mathcal{G}}$ is Cohen-Macaulay whenever \mathbb{k} is;

- (c) $\mathbb{k}[L]^\mathcal{G}$ is Cohen-Macaulay for $\mathbb{k} = \mathbb{Z}/(|\mathcal{G}|)$;
- (d) $\mathbb{F}_p[L]^\mathcal{G}$ is Cohen-Macaulay for all primes p dividing $|\mathcal{G}|$.

Proof. (a) \Rightarrow (b): Put $S = \mathbb{k}[L]^\mathcal{G}$ and consider the extension of rings $\mathbb{k} \hookrightarrow S$. This extension is free; see §2.1. By [BH, Exercise 2.1.23], S is Cohen-Macaulay if (and only if) \mathbb{k} is Cohen-Macaulay and, for all $\mathfrak{P} \in \text{Spec } S$, the fibre $S_{\mathfrak{P}}/\mathfrak{p}S_{\mathfrak{P}}$ is Cohen-Macaulay, where $\mathfrak{p} = \mathfrak{P} \cap \mathbb{k}$. But $S_{\mathfrak{P}}/\mathfrak{p}S_{\mathfrak{P}}$ is a localization of $(S/\mathfrak{p}S)_{\mathfrak{p} \setminus 0} \cong Q(\mathbb{k}/\mathfrak{p})[L]^\mathcal{G}$; see equation (2.1). Therefore, by [BH, Theorem 2.1.3(b)], it suffices to show that $Q(\mathbb{k}/\mathfrak{p})[L]^\mathcal{G}$ is Cohen-Macaulay. In other words, we may assume that \mathbb{k} is a field. By [BH, Theorem 2.1.10], we may further assume that $\mathbb{k} = \mathbb{Q}$ or $\mathbb{k} = \mathbb{F}_p$. But equation (2.1) implies that $\mathbb{Q}[L]^\mathcal{G} = \mathbb{Z}[L]_{\mathbb{Z} \setminus 0}^\mathcal{G}$ and $\mathbb{F}_p[L]^\mathcal{G} \cong \mathbb{Z}[L]^\mathcal{G}/(p)$. Since $\mathbb{Z}[L]^\mathcal{G}$ is assumed Cohen-Macaulay, [BH, Theorem 2.1.3] implies that $\mathbb{Q}[L]^\mathcal{G}$ and $\mathbb{F}_p[L]^\mathcal{G}$ are Cohen-Macaulay, as desired.

(b) \Rightarrow (c) is clear.

(c) \Rightarrow (d): Write $|\mathcal{G}| = \prod_p p^{n_p}$. Then $\mathbb{k}[L] \cong \prod_p \mathbb{Z}/(p^{n_p})[L]^\mathcal{G}$ and $\mathbb{Z}/(p^{n_p})[L]^\mathcal{G}$ is a localization of $\mathbb{k}[L]^\mathcal{G}$. Therefore, $\mathbb{Z}/(p^{n_p})[L]^\mathcal{G}$ is Cohen-Macaulay, by [BH, Theorem 2.1.3(b)]. If $n_p \neq 0$ then it follows from [BH, Theorem 2.1.3(a)] that $\mathbb{Z}_{(p)}[L]^\mathcal{G}$ and $\mathbb{F}_p[L]^\mathcal{G} \cong \mathbb{Z}_{(p)}[L]^\mathcal{G}/(p)$ are Cohen-Macaulay.

(d) \Rightarrow (a): First, (d) implies that $\mathbb{F}_p[L]^\mathcal{G}$ is Cohen-Macaulay for all primes p . For, if p does not divide $|\mathcal{G}|$ then $\mathbb{F}_p[L]^\mathcal{G}$ is always Cohen-Macaulay; see [BH, Corollary 6.4.6]. Now let \mathfrak{P} be a maximal ideal of $\mathbb{Z}[L]$. Then $\mathfrak{P} \cap \mathbb{Z} = (p)$ for some prime p and $\mathbb{Z}[L]_{\mathfrak{P}}^\mathcal{G}/(p)$ is a localization of $\mathbb{Z}[L]^\mathcal{G}/(p) = \mathbb{F}_p[L]^\mathcal{G}$. Thus, $\mathbb{Z}[L]_{\mathfrak{P}}^\mathcal{G}/(p)$ is Cohen-Macaulay and [BH, Theorem 2.1.3(a)] further implies that $\mathbb{Z}[L]_{\mathfrak{P}}^\mathcal{G}$ is Cohen-Macaulay. Since, \mathfrak{P} was arbitrary, (a) is proved. \square

Since normal rings of (Krull) dimension at most 2 are Cohen-Macaulay, the implication (d) \Rightarrow (b) of the lemma shows that $\mathbb{k}[L]^\mathcal{G}$ is certainly Cohen-Macaulay whenever \mathbb{k} is Cohen-Macaulay and L has rank at most 2.

3.3. Proof of the Theorem. We are now ready to prove the Theorem stated in the Introduction. Recall that, for any subgroup $\mathcal{H} \leq \mathcal{G}$, $\mathcal{H}^{(2)}$ denotes the subgroup generated by the elements of \mathcal{H} that act as bireflections on L or, equivalently, by the subgroups of \mathcal{H} that belong to \mathfrak{X}_2 ; see (2.2). Throughout, we assume that $\mathbb{Z}[L]^\mathcal{G}$ is Cohen-Macaulay.

We first show that $\mathcal{G}_m/\mathcal{G}_m^{(2)}$ is a perfect group for all $m \in L$. Put $\mathbb{k} = \mathbb{Z}/(|\mathcal{G}|)$. Then $\mathbb{k}[L]^\mathcal{G}$ is Cohen-Macaulay, by Lemma 3.2. Therefore, the restriction $H^1(\mathcal{G}, \mathbb{k}[L]) \rightarrow \prod_{\mathcal{H} \in \mathfrak{X}_2} H^1(\mathcal{H}, \mathbb{k}[L])$ is injective, by Proposition 1.4; see the remark in §3.1. Corollary 2.5 yields that all $\mathcal{G}_m^{\text{ab}}$ are generated by the images of the subgroups $\mathcal{H} \leq \mathcal{G}_m$ with $\mathcal{H} \in \mathfrak{X}_2$. In other words,

each $\mathcal{G}_m^{\text{ab}}$ is generated by the images of the bireflections in \mathcal{G}_m . Therefore, $(\mathcal{G}_m/\mathcal{G}_m^{(2)})^{\text{ab}} = 1$, as desired.

Now assume that \mathcal{G} acts non-trivially on L . Our goal is to show that some isotropy group \mathcal{G}_m is non-perfect. Suppose otherwise. Replacing \mathcal{G} by $\mathcal{G}/\text{Ker}_{\mathcal{G}}(L)$ we may assume that $1 \neq \mathcal{G}$ acts faithfully on L . Then $\mathfrak{X}_k = \{1\}$ for all $k < 8$, by Proposition 2.4. It follows that

$$k = \inf\{i > 0 \mid H^i(\mathcal{G}, \mathbb{k}[L]) \neq 0\} \geq 7.$$

Indeed, if $k < 7$ then Proposition 1.4 implies that $0 \neq H^k(\mathcal{G}, \mathbb{k}[L])$ embeds into $\prod_{\mathcal{H} \in \mathfrak{X}_{k+1}} H^k(\mathcal{H}, \mathbb{k}[L])$ which is trivial, because $\mathfrak{X}_{k+1} = \{1\}$. By Lemma 2.5 with $\mathfrak{X} = \{1\}$, we conclude that

$$H^i(\mathcal{G}_m, \mathbb{k}) = 0 \text{ for all } m \in L \text{ and all } 0 < i < 7.$$

On the other hand, choosing \mathcal{G}_m minimal with $\mathcal{G}_m \neq 1$, we know by Lemmas 2.3 and 2.4(b) that \mathcal{G}_m is isomorphic to the binary icosahedral group $2.\mathcal{A}_5$. The cohomology of $2.\mathcal{A}_5$ is 4-periodic (see [Br, p. 155]). Hence, $H^3(\mathcal{G}_m, \mathbb{k}) \cong H^{-1}(\mathcal{G}_m, \mathbb{k}) = \text{ann}_{\mathbb{k}}(\sum_{\mathcal{G}_m} g) \cong \mathbb{Z}/(|\mathcal{G}_m|) \neq 0$. This contradiction completes the proof of the Theorem. \square

3.4. Rational invariance. We now show that the Cohen-Macaulay property of $\mathbb{k}[L]^{\mathcal{G}}$ depends only on the rational isomorphism class of the \mathcal{G} -lattice L . Recall that \mathcal{G} -lattices L and L' are said to be *rationally isomorphic* if $L \otimes_{\mathbb{Z}} \mathbb{Q} \cong L' \otimes_{\mathbb{Z}} \mathbb{Q}$ as $\mathbb{Q}[\mathcal{G}]$ -modules. In this section, \mathbb{k} denotes any commutative base ring.

Proposition. *If $\mathbb{k}[L]^{\mathcal{G}}$ is Cohen-Macaulay then so is $\mathbb{k}[L']^{\mathcal{G}}$ for any \mathcal{G} -lattice L' that is rationally isomorphic to L .*

Proof. Assume that $L \otimes_{\mathbb{Z}} \mathbb{Q} \cong L' \otimes_{\mathbb{Z}} \mathbb{Q}$. Replacing L' by an isomorphic copy inside $L \otimes_{\mathbb{Z}} \mathbb{Q}$, we may assume that $L \supseteq L'$ and L/L' is finite. Then $\mathbb{k}[L]$ is finite over $\mathbb{k}[L']$ which in turn is integral over $\mathbb{k}[L']^{\mathcal{G}}$. Therefore, $\mathbb{k}[L]$ is integral over $\mathbb{k}[L']^{\mathcal{G}}$, and hence so is $\mathbb{k}[L]^{\mathcal{G}}$.

We now invoke a ring-theoretic result of Hochster and Eagon [HE] (or see [BH, Theorem 6.4.5]): Let $R \supseteq S$ be an integral extension of commutative rings having a Reynolds operator, that is, an S -linear map $R \rightarrow S$ that restricts to the identity on S . If R is Cohen-Macaulay then S is Cohen-Macaulay as well.

To construct the requisite Reynolds operator, consider the truncation map

$$\pi: \mathbb{k}[L] \rightarrow \mathbb{k}[L'], \quad \sum_{m \in L} k_m \mathbf{x}^m \mapsto \sum_{m \in L'} k_m \mathbf{x}^m.$$

This is a Reynolds operator for the extension $\mathbb{k}[L] \supseteq \mathbb{k}[L']$ that satisfies $\pi(g(f)) = g(\pi(f))$ for all $g \in \mathcal{G}$, $f \in \mathbb{k}[L]$. Therefore, π restricts to a Reynolds operator $\mathbb{k}[L]^\mathcal{G} \rightarrow \mathbb{k}[L']^\mathcal{G}$ and the proposition follows. \square

The proposition in particular allows to reduce the general case of the Cohen-Macaulay problem for multiplicative invariants to the case of effective \mathcal{G} -lattices. Recall that the \mathcal{G} -lattice L is *effective* if $L^\mathcal{G} = 0$. For any \mathcal{G} -lattice L , the quotient $L/L^\mathcal{G}$ is an effective \mathcal{G} -lattice; this follows, for example, from the fact that L is rationally isomorphic to the \mathcal{G} -lattice $L^\mathcal{G} \oplus L/L^\mathcal{G}$.

Corollary. $\mathbb{k}[L]^\mathcal{G}$ is Cohen-Macaulay if and only if this holds for $\mathbb{k}[L/L^\mathcal{G}]^\mathcal{G}$.

Proof. By the proposition, we may replace L by $L' = L^\mathcal{G} \oplus L/L^\mathcal{G}$. But $\mathbb{k}[L']^\mathcal{G} \cong \mathbb{k}[L/L^\mathcal{G}]^\mathcal{G} \otimes_{\mathbb{k}} \mathbb{k}[L^\mathcal{G}]^\mathcal{G}$, a Laurent polynomial algebra over $\mathbb{k}[L/L^\mathcal{G}]^\mathcal{G}$. Thus, by [BH, Theorems 2.1.3 and 2.1.9], $\mathbb{k}[L']^\mathcal{G}$ is Cohen-Macaulay if and only if $\mathbb{k}[L/L^\mathcal{G}]^\mathcal{G}$ is Cohen-Macaulay. The corollary follows. \square

3.5. Remarks and examples.

3.5.1. *Abelian bireflection groups.* It is not hard to show that if \mathcal{G} is a finite abelian group acting as a bireflection group on the lattice L then $\mathbb{Z}[L]^\mathcal{G}$ is Cohen-Macaulay. Using Corollary 3.4 and an induction on rank L , the proof reduces to the verification that $\mathbb{Z}[L]^\mathcal{G}$ is Cohen-Macaulay for $L = \mathbb{Z}^n$ and $\mathcal{G} = \text{diag}(\pm 1, \dots, \pm 1) \cap \text{SL}_n(\mathbb{Z})$. Direct computation shows that, for $n \geq 2$,

$$\mathbb{Z}[L]^\mathcal{G} = \mathbb{Z}[\xi_1, \dots, \xi_n] \oplus \eta \mathbb{Z}[\xi_1, \dots, \xi_n]$$

where $\xi_i = \mathbf{x}^{e_i} + \mathbf{x}^{-e_i}$ is the \mathcal{G} -orbit sum of the standard basis element $e_i \in \mathbb{Z}^n$ and η is the orbit sum of $\sum_i e_i = (1, \dots, 1)$.

It would be worthwhile to try and extend this result to larger classes of bireflection groups. The aforementioned classification of bireflection groups in [GuS] will presumably be helpful in this endeavor.

3.5.2. *Subgroups of reflection groups.* Assume that \mathcal{G} acts as a reflection group on the lattice L and let \mathcal{H} be a subgroup of \mathcal{G} with $[\mathcal{G} : \mathcal{H}] = 2$. Then \mathcal{H} acts as a bireflection group. (More generally, if \mathcal{G} acts as a k -reflection group and $[\mathcal{G} : \mathcal{H}] = m$ then \mathcal{H} acts as a km -reflection group; see [L₁].) Presumably $\mathbb{Z}[L]^\mathcal{H}$ will always be Cohen-Macaulay, but I have no proof. For an explicit example, let $\mathcal{G} = \mathcal{S}_n$ be the symmetric group on $\{1, \dots, n\}$ and let $L = U_n$ be the standard permutation lattice for \mathcal{S}_n ; so $U_n = \bigoplus_{i=1}^n \mathbb{Z}e_i$ with $s(e_i) = e_{s(i)}$ for $s \in \mathcal{S}_n$. Transpositions act as reflections on U_n and 3-cycles as bireflections. Let $\mathcal{A}_n \leq \mathcal{S}_n$ denote the alternating group. To compute $\mathbb{Z}[U_n]^{\mathcal{A}_n}$, put $x_i = \mathbf{x}^{e_i} \in \mathbb{Z}[U_n]$. Then $\mathbb{Z}[U_n] = \mathbb{Z}[x_1, \dots, x_n][s_n^{-1}]$, where $s_n = \mathbf{x}^{\sum_1^n e_i} = \prod_1^n x_i$ is the n^{th} elementary symmetric function, and \mathcal{S}_n acts via $s(x_i) = x_{s(i)}$ ($s \in \mathcal{S}_n$).

Therefore, $\mathbb{Z}[U_n]^{\mathcal{A}_n} = \mathbb{Z}[x_1, \dots, x_n]^{\mathcal{A}_n}[s_n^{-1}]$. The ring $\mathbb{Z}[x_1, \dots, x_n]^{\mathcal{A}_n}$ of polynomial \mathcal{A}_n -invariants has the following form; see [S, Theorem 1.3.5]: $\mathbb{Z}[x_1, \dots, x_n]^{\mathcal{A}_n} = \mathbb{Z}[s_1, \dots, s_n] \oplus d\mathbb{Z}[s_1, \dots, s_n]$, where s_i is the i^{th} elementary symmetric function and

$$d = \frac{1}{2}(\Delta + \Delta_+)$$

with $\Delta_+ = \prod_{i < j} (x_i + x_j)$ and $\Delta = \prod_{i < j} (x_i - x_j)$, the Vandermonde determinant. Thus,

$$\mathbb{Z}[U_n]^{\mathcal{A}_n} = \mathbb{Z}[s_1, \dots, s_{n-1}, s_n^{\pm 1}] \oplus d\mathbb{Z}[s_1, \dots, s_{n-1}, s_n^{\pm 1}]$$

This is Cohen-Macaulay, being free over $\mathbb{Z}[s_1, \dots, s_{n-1}, s_n^{\pm 1}]$.

3.5.3. \mathcal{S}_n -lattices. If L is a lattice for the symmetric group \mathcal{S}_n such that $\mathbb{Z}[L]^{\mathcal{S}_n}$ is Cohen-Macaulay then the Theorem implies that \mathcal{S}_n acts as a bireflection group on L , and hence on all simple constituents of $L \otimes_{\mathbb{Z}} \mathbb{Q}$. The simple $\mathbb{Q}[\mathcal{S}_n]$ -modules are the Specht modules S^λ for partitions λ of n . If $n \geq 7$ then the only partitions λ so that \mathcal{S}_n acts as a bireflection group on S^λ are (n) , (1^n) and $(n-1, 1)$; this follows from the lists in [Hu] and [W]. The corresponding Specht modules are trivial module, \mathbb{Q} , the sign module \mathbb{Q}^- , and the rational root module $A_{n-1} \otimes_{\mathbb{Z}} \mathbb{Q}$, where $A_{n-1} = \{\sum_i z_i e_i \in U_n \mid \sum_i z_i = 0\}$ and U_n is as in §3.5.2. Thus, if $n \geq 7$ and $\mathbb{Z}[L]^{\mathcal{S}_n}$ is Cohen-Macaulay then we must have

$$L \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathbb{Q}^r \oplus (\mathbb{Q}^-)^s \oplus (A_{n-1} \otimes_{\mathbb{Z}} \mathbb{Q})^t$$

with $s + t \leq 2$. In most cases, $\mathbb{Z}[L]^{\mathcal{S}_n}$ is easily seen to be Cohen-Macaulay. Indeed, we may assume $r = 0$ by Corollary 3.4. If $s + t \leq 1$ then \mathcal{S}_n acts as a reflection group on L and so $\mathbb{Z}[L]^{\mathcal{S}_n}$ is Cohen-Macaulay by [L₂]. For $t = 0$ we may quote the last remark in §3.2. This leaves the cases $s = t = 1$ and $s = 0, t = 2$ to consider.

If $s = t = 1$ then add a copy of \mathbb{Q} so that L is rationally isomorphic to $U_n \oplus \mathbb{Z}^-$. Using the notation of §3.5.2 and putting $t = \mathbf{x}^{(0U_n, 1)} \in \mathbb{Z}[U_n \oplus \mathbb{Z}^-]$ the invariants are:

$$\mathbb{Z}[U_n \oplus \mathbb{Z}^-]^{\mathcal{S}_n} = R \oplus R\varphi$$

with $R = \mathbb{Z}[s_1, \dots, s_{n-1}, s_n^{\pm 1}, t + t^{-1}]$ and $\varphi = \frac{1}{2}(\Delta_+ + \Delta)t + \frac{1}{2}(\Delta_+ - \Delta)t^{-1}$.

If $s = 0$ and $t = 2$ then we may replace L by the lattice $U_n^2 = U_n \oplus U_n$. By Lemma 3.2 $\mathbb{Z}[U_n^2]^{\mathcal{S}_n}$ is Cohen-Macaulay precisely if $\mathbb{F}_p[U_n^2]^{\mathcal{S}_n}$ is Cohen-Macaulay for all primes $p \leq n$. As in §3.5.2, one sees that $\mathbb{F}_p[U_n^2]^{\mathcal{S}_n}$ is a localization of the algebra “vector invariants” $\mathbb{F}_p[x_1, \dots, x_n, y_1, \dots, y_n]^{\mathcal{S}_n}$. By [K₂, Corollary 3.5], this algebra is known to be Cohen-Macaulay for $n/2 < p \leq n$, but the primes $p \leq n/2$ apparently remain to be dealt with.

3.5.4. *Ranks* ≤ 4 . As was pointed out in §3.2, $\mathbb{Z}[L]^{\mathcal{G}}$ is always Cohen-Macaulay when $\text{rank } L \leq 2$.

For $L = \mathbb{Z}^3$, there are 32 \mathbb{Q} -classes of finite subgroups $\mathcal{G} \leq \text{GL}_3(\mathbb{Z})$. All \mathcal{G} are solvable; in fact, their orders divide 48. The Sylow 3-subgroup $\mathcal{H} \leq \mathcal{G}$, if nontrivial, is generated by a bireflection of order 3. Thus, $\mathbb{F}_3[L]^{\mathcal{H}}$ is Cohen-Macaulay, and hence so is $\mathbb{F}_3[L]^{\mathcal{G}}$. Therefore, by Lemma 3.2, $\mathbb{Z}[L]^{\mathcal{G}}$ is Cohen-Macaulay if and only if $\mathbb{F}_2[L]^{\mathcal{G}}$ is Cohen-Macaulay, and for this to occur, \mathcal{G} must be generated by bireflections. It turns out that 3 of the 32 \mathbb{Q} -classes consist of non-bireflection groups; these classes are represented by the cyclic groups

$$\left\langle \begin{pmatrix} -1 & & \\ & -1 & \\ & & -1 \end{pmatrix} \right\rangle, \quad \left\langle \begin{pmatrix} -1 & & \\ & 1 & \\ & & -1 \end{pmatrix} \right\rangle, \quad \left\langle \begin{pmatrix} & & -1 \\ & -1 & \\ & & -1 \end{pmatrix} \right\rangle$$

of orders 2, 4 and 6 (the latter two classes each split into 2 \mathbb{Z} -classes). For the \mathbb{Q} -classes consisting of bireflection groups, Pathak [Pk] has checked explicitly that $\mathbb{F}_2[L]^{\mathcal{G}}$ is indeed Cohen-Macaulay.

In rank 4, there are 227 \mathbb{Q} -classes of finite subgroups $\mathcal{G} \leq \text{GL}_4(\mathbb{Z})$. All but 5 of them consist of solvable groups and 4 of the non-solvable classes are bireflection groups, the one exception being represented by \mathcal{S}_5 acting on the signed root lattice $\mathbb{Z}^- \otimes_{\mathbb{Z}} A_4$. Thus, if the group $\mathcal{G}/\mathcal{G}^{(2)}$ is perfect then it is actually trivial, that is, \mathcal{G} is a bireflection group. It also turns out that, in this case, all isotropy groups \mathcal{G}_m are bireflection groups. There are exactly 71 \mathbb{Q} -classes that do not consist of bireflection groups. By the foregoing, they lead to non-Cohen-Macaulay multiplicative invariant algebras. The \mathbb{Q} -classes consisting of bireflection groups have not been systematically investigated yet. The searches in rank 4 were done with [GAP].

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