8023 – Numerical Differential Equations – HW4.

Due date: 10 Mar 2009. Late assignments will not be accepted.

1) Let $\hat{\phi}_1 = 1 - x - y$, $\hat{\phi}_2 = x$, $\hat{\phi}_3 = y$ be the usual basis function for the piecewise linear elements on the usual reference element $\hat{K}$ (a triangle with vertices at $(0,0)$, $(1,0)$ and $(0,1)$). We now build a basis for some piecewise linear elements in $H(div)$, and compute the stiffness matrix. The basis for V_h is given by pairs (ϕ_i, ϕ_j) and each such pair is labelled ψ_k. Hence, we have $\hat{\psi}_1, \ldots, \hat{\psi}_9)$; i.e., there are 9 basis functions.

a) Write the 9 basis functions explicitly, in the following order: $\hat{\psi}_1 = (\hat{\phi}_1, \hat{\phi}_1)$, $\hat{\psi}_2 = (\hat{\phi}_1, \hat{\phi}_2)$, ..., $\hat{\psi}_9 = (\hat{\phi}_3, \hat{\phi}_3)$. Example: $\hat{\psi}_1 = (1 - x - y, 1 - x - y)$, etc...

b) Compute the matrix A whose entry A_{ij} is $\int_{\hat{K}} \hat{\psi}_i^T \hat{\psi}_j$. This is a 9×9 symmetric matrix. Your answer should take the form of a 9×9 matrix. You may use MAPLE or other computer algebra system to speed up your calculations, and I don't need to see intermediate steps, just the final 9×9 matrix is good enough. But make sure it's right!

2) The GL nodes and weights are defined by $x_0 = -1, x_n = 1$ and the relations $\int_{-1}^{1} x^k = \sum_{j=0}^{n} w_j x_j^k$ with $k = 0, 1, \ldots, 2n - 1$.

a) Write all the relations for $n = 3$, and compute the integrals explicitly.

b) Find the GL nodes and weights for $n = 3$. (Hint: your life will be simpler if you note that $x_1 = -x_2$, $w_0 = w_3$ and $w_1 = w_2$ takes care of odd values of k and halves the number of unknowns.)

3) In this problem, we experimentally measure the second order convergence rate of the FEM. We again use the mesh you created for the first letter of your first name. Normalize your mesh, so that $\min(v(:,j)) = 0$ and $\max(v(:,j)) = 1$, $j = 1, 2$. I will be able to see whether you did this right in the plots you hand in. We will be solving the problem

$$\begin{cases} -\Delta u &= g \text{ in } \Omega, \\ u &= h \text{ on } \partial\Omega. \end{cases}$$

We will be using the known true solution $u(x, y) = \sin(20((x - 1/4)^2 + (y - 1/2)^2))$.

a) Find g for our true solution. Hand in (handwritten) the formula that you obtain (I don't need the code.)

b) Subdivide your grid suitably many times (you should have that the grid interval `delta = norm(v(f(1,1),:)-v(f(1,2),:))` is 0.05 or less). Produce a four-paneled plot as follows: in the upper-left corner, put the true solution. In the lower-left corner, put the computed solution. In the upper-right corner, put the right-hand-side g. In the bottom-right corner, put the error $u_\delta - u_{true}$. Use `colorbar` *for each of the 4 plots*, so that we see the scales are correct. Use `subplot` to create the 4 panels. Use `title` to ensure that I can tell which plot is which. Hand in this plot.

c) Solve now the problem for finer and finer subdivision. At each step k, store the grid parameter in `delta(k)`, and store the L^2 norm of the error in `L2(k)`. The L^2 norm of the error is approximately `sqrt(error'*M*error)`. Create a `loglog` plot of δ against L^2 error. Use a solid line with square markers for the measured error (i.e., use `loglog(delta,L2,'k-s')`). On the same plot, include the predicted δ^2 order of convergence, using a dotted line (e.g., `'k:s'`). Hand in this plot. Use `hold on` to put two graphs on the same plot.