8023 – Numerical Differential Equations – HW3.

Due date: 24 Feb 2009. Late assignments will not be accepted.

1) Let A be a real symmetric matrix. We denote by $|u|$ the Euclidian norm of u and by $\|A\|_2$ the matrix norm $\max_{u\neq 0}|Au|/|u| = \max_{|u|=1}|Au|$ of A, and the matrix norm $\|A\|_a$ is $\max_{u\neq 0}|Au|_a/|u|_a = \max_{|u|_a=1}|Au|_a$, where $|\cdot|_a$ is some vector norm. The condition number $\chi_a(A)$ is defined as $\|A\|_a\|A^{-1}\|_a$. A is Positive Definite (PD) if $u^TAu > 0$ for all nonzero u.

a) State the spectral theorem for real symmetric matrices. (Look it up if you need to.)

For the rest of this problem, always work from *first principles*. You can use Cauchy-Schwarz, and results we proved in class, but don't use Jordan forms or Singular Value Decompositions, or handwavy arguments about norms.

b) If A is Symmetric Positive Definite (SPD), find an SPD matrix $A^{1/2}$, such that $(A^{1/2})^2 = A$. (Hint: use the spectral theorem. You will need to show that A is PD iff $\lambda_{\min} > 0$.)

c) Show that $\|A\|_2 = \max_{|u|=|v|=1} u^TAv$. (Hint: prove that $|v| = w^Tv$ where $|w| = 1$ [what's w?]; apply this to $\|A\|_2 = |Au_{\max}|$. Conversely, $v^TAu \overset{\text{C-S}}{\leq} |v||Au| \leq ...$)

d) If A is real symmetric, $\|A\|_2 = \max_{|u|=1}|u^TAu|$. (Hint: first assume $A = D$ is diagonal, and prove that $\max_{|u|=|v|=1} v^TDu = |\lambda_{\max}| = \max_{|u|=1}|u^TDu|$. Don't wave your hands. Then use the spectral theorem. Don't wave your hands.)

e) Let $a(u,v)$ be an inner product on $\mathbb{R}^n$. Find a matrix A for which $a(u,v) = u^TAv$, and show that A is SPD. (Hint: Show that $A_{ij} = a(e_i, e_j)$.)

f) Let B be a matrix such that $a(Bu,v) = a(u,Bv)$ for every u, v. Show that $\|B\|_a = \max_{a(u,u)=1}|a(Bu,u)|$. (Hint: Use the change of variables $v = A^{-1/2}u$ to show that $\|B\|_a = \|A^{1/2}BA^{-1/2}\|_2$. Show that $A^{1/2}BA^{-1/2}$ is symmetric. Reverse the change of variables.)

g) We analyzed in class the Additive Schwarz preconditioner, the analysis follows Toselli & Widlund. Our Lemma 2.5 says (something like) $a(P_{ad}^{-1}u,u) \leq C_0^2 a(u,u)$. Our Lemma 2.6 says that $a(P_{ad}u,u) \leq (\rho(\epsilon)+1)a(u,u)$. We also showed in an unnumbered Lemma after Lemma 2.5, that $a(P_{ad}u,v) = a(u,P_{ad}v)$. We also have[1] that $a(P_{as}^{\pm 1}u,u) \geq 0$. I concluded by giving an upper bound for χ_{sp}, this was wrong. Considering parts b) through f), give the correct upper bound for the correct condition number.

2) In this problem, we complete our FEM solver for { $-\Delta u = g$ in Ω, and $u = 0$ on $\partial\Omega$ }.

a) Assemble the Stiffness Matrix A from the previous homework, and the Mass Matrix M in a similar way. Put this code in a function `[A,M]=laplacian_bilinear(f,v)`. Print out this code and hand it in.

b) Put the code for finding the boundary vertices and the interior vertices in a function `[interior,boundary]=find_boundary(f,v)`. Print out this code and hand it in.

The rest of this problem uses the mesh of the first letter of your first name. You should subdivide it 4-5 times to get a good precision.

c) Choose a function g for the right-hand-side of the PDE; a heat source located somewhere in a corner of the first letter of your first name. Plot this function and hand this plot in.

d) Finally, compute the FEM solution u with your RHS g. Plot the solution u, as well as its logarithm $\log u$ (assuming $g \geq 0$ then also $u \geq 0$). Label each plot properly and hand them in.

[1] This is implied by Lemmas 2.5 and 2.6.