# 8023 – Numerical Differential Equations – HW2.

Due date: 10 Feb 2009. Late assignments will not be accepted.

1) Let $A$ be a real symmetric matrix. We denote by $|u|$ the Euclidian norm of $u$ and by $\|A\|_2$ the matrix norm $\max_{u \neq 0} |Au|/|u|$ of $A$.

a) State the spectral theorem for real symmetric matrices. (Look it up if you need to.)

b) If $A$ is Symmetric Positive Definite (SPD), find an SPD matrix $A^{1/2}$, such that $(A^{1/2})^2 = A$.

c) Show that $\|A\|_2 = \max_{|u|=|v|=1} u^T Av$ and, if $A$ is SPD, $\|A\|_2 = \max_{|u|=1} u^T Au$. (Remark: the second equality is false for $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, but the first equality is true even if $A$ is nonsymmetric.)

e) Let $a(u, v)$ be an inner product on $\mathbb{R}^n$. Find a matrix $A$ for which $a(u, v) = u^T Av$, and show that $A$ is SPD.

e) Let $B$ be a matrix such that $a(Bu, v) = a(u, Bv)$ for every $u, v$. Show that $\|B\|_a = \max_{a(u,u)=1} a(Bu, u)$. (Hint: show that $\|B\|_a = \|A^{1/2} BA^{-1/2}\|_2$ and that $A^{1/2} BA^{-1/2}$ is symmetric. As usual, $\|B\|_a = \max_{a(u,u)=1} \sqrt{a(Bu, Bu)}$.)

f) We analyzed in class the Additive Schwarz preconditioner and gave the estimate $\chi_{sp}(P_{ad}) \leq C_0^2(\rho(\epsilon) + 1)$. The upper bound is correct, but the $\chi_{sp}$ is incorrect. What should I have written instead of $\chi_{sp}$?

2) In this exercise, we implement the first portion of our Finite Element code, which computes the large $B$ matrix and manipulates it. Let `v = [0 0; 1 0; 1 1; 0 1];` and `f = [1 2 3 ; 1 3 4];` (that's the unit square with two triangles). For each item in the following list, hand in the result of the code on this unit square.

a) Compute the $U$ vectors using a single vectorized line. (Hint: `U = v(f(:,2),:)  - v(f(:,1),:);`)

b) Compute the $V$ vectors using a single vectorized line.

c) Compute simultaneously all the $B$ matrices, one matrix per row. (Hint: `B = [U V];`).

d) Compute simultaneously all the determinants: `detB = ...`

e) Compute simultaneously all the inverses: `invB = ...`

f) Let `p = [0 1];` and `q = [0 1];`. Calculate $|\det(B)|(B^{-1}p^T) \cdot (B^{-1}q^T)$, where the dot denotes the dot product. Compute this number for each of the matrices $B$ in our $n \times 4$ array `B`, and hand it in. Use vectorized code, not looping! For our square, there are two $2 \times 2$ matrices $B$, stored as rows of the array `B`, which is $2 \times 4$. Since `B` is stored in a weird way, you have to write this carefully. Since each matrix $B$ produces one scalar output, the output should be a $2 \times 1$ vector. (Hint: you may find `dot(...,...,2)` useful.)

Finally, after you have executed your code line by line on the sample input (the square) and printed out all the output, put all your code in a file called `compute_entries.m`, put `function entries = compute_entries(f,v,p,q)` at the top. Hand in this code. Also hand in the result of `compute_entries(f,v,[0 1],[1 0])`, where `f` and `v` make up the mesh for the first letter of your first name, like in the first homework. (Don't subdivide, or it'll be too big.)