

## 8023 – Numerical Differential Equations – HW1.

Due date: 3 Feb 2009. Late assignments will not be accepted.

1) Let  $\Omega = (0, \pi) \times (0, \infty)$ ,  $\Gamma = ]0, \pi[ \times \{0\}$ ,  $u_k = e^{-ky} \sin(kx)$  for  $k = 1, 2, \dots$

a) Show that  $u_k$  solves

$$\begin{cases} \frac{\partial^2}{\partial x^2} u_k + \frac{\partial^2}{\partial y^2} u_k = 0 & \text{in } \Omega, \\ u_k(0, y) = u_k(\pi, y) = 0 & \forall y > 0, \\ u_k(x, 0) = \sin kx & \text{on } \Gamma. \end{cases}$$

We will see later that this equivalent to finding  $u_k$  that solves the problem

$$\begin{cases} \text{Find } v \in H^1(\Omega) \text{ minimizing } \int_{\Omega} (\nabla v)^2 \text{ such that} \\ v(x, 0) = \sin(kx) \text{ on } \Gamma \text{ and } v(0, y) = v(\pi, y) = 0 \quad \forall y > 0. \end{cases} \quad (1)$$

b) Calculate  $\int_{\Omega} (\nabla u_k)^2$ .

c) Using the “Fourier” formula  $\|\sin(kx)\|_{H^s(]0, \pi])} = |k|^s$ , find the smallest value of  $s = s_0$  for which

$$\|u_k\|_{H^1(\Omega)} \leq C_s \|u_k\|_{H^s(\Gamma)} \text{ for } k = 1, 2, \dots$$

Calculate  $C_s$ .

d) Show that there is no continuous “prolongation” linear operator  $R : H^s(\Gamma) \rightarrow \{v \in H^1(\Omega) : v(0, y) = v(\pi, y) = 0\}$ , whenever  $s < s_0$ . (Hint: use (1).) How does this compare with the Trace Theorem?

2) Let  $A$  be the  $1024 \times 1024$  matrix  $A = \text{tridiag}(-I, E, -I)$ , i.e.,  $A$  is a block-tridiagonal matrix with the block  $E$  on its diagonal, and blocks  $-I$  above and below the diagonal. The matrix  $E$  is  $32 \times 32$  and  $E = \text{tridiag}(-1, 4, -1)$  (it has fours on the diagonal).

Find the eigenvectors corresponding to the 5 smallest eigenvalues. Use `surf(reshape(u,32,32))` to visualize them (they should be smooth). Hand in the 5 graphs. (Hint: if it's too slow, make sure that you are using sparse matrices. Use “`eigs(A,5,'sm')`” for the smallest eigenvalues.)

3) Implement “`subdividemesh.m`”. Hand in a hard copy of this code. Create a mesh with the first letter of your first name. Subdivide the letter twice. Plot the image as we did in class, hand in this plot. Plot the same mesh again with random errors added to the vertices, to make sure that your vertices are shared. Hand in this plot.