GOLDIE RANKS OF SKEW POWER SERIES RINGS OF AUTOMORPHIC TYPE

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ABSTRACT. Let A be a noetherian ring equipped with an automorphism α, and let $B := A[[y;\alpha]]$ denote the corresponding skew power series ring. We prove that A is semiprime if and only if B is semiprime. Next, assuming A is semiprime, we prove that the Goldie rank of B is equal to the Goldie rank of A. The same conclusions hold true when B is replaced by the skew Laurent series ring $A[[y^{\pm 1};\alpha]]$.

1. INTRODUCTION

This note is concerned with the Goldie ranks of skew power series rings of automorphic type (i.e., with zero skew derivation), over noetherian semiprime coefficient rings.

1.1. Studies of Goldie rank in skew polynomial extensions include [1], [3], [5], [6], [9], [10], [12], [15], and [16]. The chronology relevant to our present purposes can be briefly summarized as follows: (1) In 1972, Shock proved that if R is a ring having finite right Goldie dimension, then the right Goldie dimension of $R[x]$ is equal to the right Goldie dimension of R [15]. (2) In 1988, Grzeszczuk proved that if R is a semiprime right Goldie ring equipped with a derivation δ, then the Goldie rank of $R[x;\delta]$ is equal to the Goldie rank of R [5]. (3) In 1995, Matczuk proved that if R is a semiprime right Goldie ring equipped with an automorphism τ and τ-derivation δ, then the Goldie rank of $R[x;\tau,\delta]$ is equal to the Goldie rank of R [9]. In 2005, Leroy and Matczuk generalized this last result to the case where τ is an injective endomorphism [6].

1.2. Suitably defined skew power series rings $R[[x;\tau,\delta]]$ were recently introduced by Venjakob [19], and subsequently studied in [13], [14], and [20]. Recent studies of skew power series rings $R[[x;\tau]]$, of *automorphic type*, include [7], [17], and [18]. Our aim in this note is to initiate the study of Goldie ranks of skew power series rings – beginning with extensions of automorphic type.

1.3. Now let A be a right or left noetherian ring equipped with an automorphism α, and let $B := A[[y;\alpha]]$ denote the corresponding skew power series ring. In (2.9)

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we prove that if A is semiprime then B is semiprime, and if B is semiprime and A is noetherian (on both sides) then A is semiprime. We further prove in (2.9) that if A is semiprime then the Goldie rank of A is equal to the Goldie rank of B. Similar statements hold for the skew Laurent series ring $A[[y^{\pm 1};\alpha]]$, obtained by localizing B at powers of y.

1.4. We also consider α-primality in (2.9), and we consider induced ideals in (2.1) and (2.10). Induced ideals appear both in a key lemma for our main results and in an application of them.

1.5. As is the case for classical commutative power series rings, our approach requires explicit use of finitely generated ideals. Consequently, it is not immediately obvious how to replace our noetherianity hypotheses with more general (e.g., Goldie) conditions.

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2. Induced Prime Ideals and Equality of Goldie Rank

Throughout, let A be a ring equipped with an automorphism α, and let $B := A[[y;\alpha]]$ denote the ring of skew power series

$$\sum_{i=0}^{\infty} a_i y^i = a_0 + a_1 y + a_2 y^2 + \cdots ,$$

for $a_0, a_1, \ldots \in A$, and with multiplication determined by $ya = \alpha(a)y$ for all $a \in A$. Of course, since α is an automorphism, we can just as well write the coefficients on the right. (By "ring" we will always mean "associative unital algebra." Also, all ring homomorphisms and all modules mentioned will be assumed to be unital.)

We first consider induced ideals, following the basic theory of commutative power series rings.

2.1. **Proposition.** *Let I be an ideal of A finitely generated on the right and left, and assume further that $\alpha(I) = I$. Then $BI = IB$, and (abusing the notation slightly),*

$$B/IB \cong (A/I)[[y;\alpha]].$$

Proof. Let $I[[y;\alpha]]$ be the set of power series in B whose coefficients, when written on the left, are all contained in I.

Assume first that $I = g_1A + \cdots + g_nA$, for some $g_1, \ldots, g_n \in I$. Choose

$$\sum_{i=0}^{\infty} h_i y^i \; \in \; I[[y;\alpha]],$$

with $h_0, h_1, \ldots \in I$. Then, for suitable choices of $r_{ij} \in A$,

$$\sum_{i=0}^{\infty} h_i y^i = \sum_{i=0}^{\infty} (g_1 r_{1i} + \cdots + g_n r_{ni}) y^i = g_1 \left(\sum_{i=0}^{\infty} r_{1i} y^i \right) + \cdots + g_n \left(\sum_{i=0}^{\infty} r_{ni} y^i \right) \in IB.$$

Hence $I[[y;\alpha]] \subseteq IB$. It is easy to see that $IB \subseteq I[[y;\alpha]]$, and so $I[[y;\alpha]] = IB$.

Since α is an automorphism and $\alpha(I) = I$, we can also view $I[[y;\alpha]]$ as the set of power series in B whose coefficients, when written on the right, are all contained in I. A mirror-image argument to the preceding one now shows that $I[[y;\alpha]] = BI$. In particular, $I[[y;\alpha]] = IB = BI$ is an ideal of B.

It is easy to see that $A[[y;\alpha]]/I[[y;\alpha]] \cong (A/I)[[y;\alpha]]$, and the proposition follows. □

2.2. Next, we briefly review some standard results concerning filtered rings and modules. The reader is referred to [8] and [11] for background.

(i) To start, let M be a right module over a ring R. Suppose further that there exist additive subgroups

$$\cdots \supseteq R_{-2} \supseteq R_{-1} \supseteq R_0 \supseteq R_1 \supseteq R_2 \supseteq \cdots$$

of R with $R_i R_j \subseteq R_{i+j}$, and additive subgroups

$$\cdots \supseteq M_{-2} \supseteq M_{-1} \supseteq M_0 \supseteq M_1 \supseteq M_2 \supseteq \cdots$$

of M with $M_i R_j \subseteq M_{i+j}$, for all integers i and j. We refer to the preceding as *filtrations* of R and M, and we include the case where $M = R$ and each $M_i = R_i$. These filtrations turn R and M into topological additive groups by letting the cosets of the R_i, for all i, form a fundamental system of neighborhoods in R, and by letting the cosets of the M_j, for all j, form a fundamental system of neighborhoods in M.

(ii) The above filtration on M is *exhaustive* if

$$M = \bigcup_{j \in \mathbb{Z}} M_j,$$

is *separated* if the intersection of the M_j is equal to zero (or equivalently, if the corresponding topology is Hausdorff), and is *complete* if Cauchy sequences converge in the corresponding topology.

(iii) The *associated graded ring* corresponding to the filtration on R is

$$\operatorname{gr} R = \cdots \oplus R_{-2}/R_{-1} \oplus R_{-1}/R_0 \oplus R_0/R_1 \oplus R_1/R_2 \oplus \cdots$$

and the *associated graded module* corresponding to the filtration on M is

$$\operatorname{gr} M = \cdots M_{-2}/M_{-1} \oplus M_{-1}/M_0 \oplus M_0/M_1 \oplus M_1/M_2 \oplus \cdots,$$

a $\operatorname{gr} R$-module.

(iv) A right R-submodule M' of M equipped with the filtration

$$M'_j := M' \cap M_j$$

is a *filtered submodule*. There is then a map $\varphi : M' \mapsto \operatorname{gr} M'$ from the set of filtered right R-submodules of M to the set of graded $\operatorname{gr} R$-submodules of $\operatorname{gr} M$. Note that φ preserves inclusions.

(v) Suppose that the filtrations on R and M are exhaustive, separated, and complete. Then M is a noetherian right R-module if $\operatorname{gr} M$ is a noetherian right $\operatorname{gr} R$-module (see, e.g., [8, pp. 60–61]).

(vi) Suppose that the filtrations on R and M are exhaustive, separated and complete. Suppose further that $\operatorname{gr} M$ is a noetherian right $\operatorname{gr} R$-module. Let M' and M'' be filtered submodules of M, and suppose that $M' \subseteq M''$. Then, following (e.g.) [8, p. 61], we find that

$$M' \subsetneq M'' \Rightarrow \varphi(M') \subsetneq \varphi(M'').$$

(vii) Of course, we have analogous results for filtered left R-modules.

2.3. Let R be a ring, let I be an ideal of R, and let M be a right R-module.

(i) We have the *I-adic filtrations*

$$R = I^0 \supseteq I^1 \supseteq \cdots$$

of R and

$$M = MI^0 \supseteq MI^1 \supseteq \cdots$$

of M. We also have the corresponding *I-adic topologies*, making R a topological ring and M a topological R-module. Note that the I-adic filtrations of R and M are exhaustive.

(ii) We also have the *Rees ring* associated to the I-adic filtration on R:

$$\widetilde{R} := R \oplus I \oplus I^2 \oplus \cdots$$

Following the definition in [8, p. 83], when I is contained in the Jacobson radical of R, and when $\widetilde{R}$ is right noetherian, R is said to be *right zariskian* with respect to the I-adic filtration.

In [8, p. 87, Proposition] it is proved that R is right zariskian with respect to the I-adic filtration if the I-adic filtration is complete and if the associated graded ring corresponding to the I-adic filtration is right noetherian.

The only consequence of the zariskian property necessary for our work below is the following: If R is right zariskian with respect to the I-adic filtration, then every right ideal of R is closed in the I-adic topology [8, p. 85, Corollary].

(iii) Again, of course, mirror-image statements to the preceding hold true when "right" is replaced by "left."

2.4. As either a left or right A-module, we can view $B = A[[y; \alpha]]$ as a direct product of copies of A, indexed by $\{0, 1, 2, \ldots\}$. Note that y is normal in B (i.e., $By = yB$), and $B/\langle y\rangle$ is naturally isomorphic to A.

(i) The $\langle y\rangle$-adic filtration on B is exhaustive, separated, and complete. The associated graded ring $\operatorname{gr} B$ is isomorphic to $A[z; \alpha]$, and so B is right (resp. left) noetherian, by (2.2v), if A is right (resp. left) noetherian.

2.5. Note that y is normal in B. We use $B' := A[[y^{\pm 1};\alpha]]$, the *skew Laurent series ring*, to denote the localization of B at powers of y.

2.6. Assume that A is right or left noetherian.

We will refer to ideals I of A such that $\alpha(I) = I$ (equivalently, $\alpha(I) \subseteq I$, by noetherianity) as *α-ideals*. Next, an α-ideal I of A (other than A itself) is *α-prime* if whenever the product of two α-ideals of A is contained in I, one of these α-ideals must itself be contained in I.

Now suppose that I is an α-prime ideal of A. It follows from [2, Remarks 4^*, 5^*, p. 338] that I is a semiprime ideal of A and is also the intersection of a finite α-orbit $I_1, \ldots, I_t$ of prime ideals in A all minimal over I.

2.7. Continue to assume that A is right or left noetherian. It is not hard to check that α extends to automorphisms of B and B' such that $\alpha(y) = y$. Consequently, we can discuss α-ideals and α-prime ideals of B and B'. Furthermore, if I is an ideal of B', then $\alpha(I) = yIy^{-1} \subseteq I$. Therefore, the α-ideals of B' are exactly the ideals of B'.

2.8. **Lemma.** *If A is right or left noetherian then:* (i) *B' is prime if and only if B' is α-prime,* (ii) *B is prime if and only if B is α-prime.*

Proof. It follows, for example, from [4, 10.18] that B is prime if and only if B' is prime, and it follows from [4, 10.18] combined with (2.6) that B is α-prime if and only if B' is α-prime. Also, it follows from (2.7) that B' is α-prime if and only if B is prime. The lemma follows. □

2.9. **Theorem.** *Assume that A is right or left noetherian.* (i) *If A is α-prime then $B = A[[y;\alpha]]$ is prime. If A is noetherian and B is prime, then A is α-prime.* (ii) *If A is semiprime then B is semiprime. If A is noetherian and B is semiprime, then A is semiprime.* (iii) *If A is semiprime then* $\operatorname{rank} A = \operatorname{rank} B$. (iv) *Statements* (i) *through* (iii) *remain true if B is replaced by $B' = A[[y^{\pm 1};\alpha]]$.*

Proof. (i) To begin, assume that A is α-prime. Let $f = f_0 + f_1 y + f_2 y^2 + \cdots$ and $g = g_0 + g_1 y + g_2 y^2 + \cdots$ be arbitrary elements of B, for $f_0, f_1, \ldots \in A$ and $g_0, g_1, \ldots \in A$. Further assume that $fB\alpha^i(g) = 0$ for all integers i. To prove that B is α-prime, it suffices to prove that f or g must equal zero. Now let m be the smallest postive integer such that $f_m \neq 0$, and let n be the smallest positive integer such that $g_n \neq 0$. It follows that $f_m A\alpha^j(g_n) = 0$, for all integers j, because $fA\alpha^i(g) = 0$ for all integers i. Since A is α-prime, it now follows that f_m or g_n is zero, a contradiction. Hence B is α-prime, and it then follows from (2.8) that B is prime.

Conversely, assume that A is noetherian and B is prime. Further assume that I and J are α-ideals of A such that $IJ = 0$. By (2.1), $BI = IB$ and $BJ = JB$ are ideals of B, and $(IB)(JB) = IJB = 0$. Thus IB or JB equals zero, since B is prime. It follows that I or J equals zero, and so A is α-prime.

(ii) Suppose that A is semiprime. To prove that B is semiprime, it suffices to prove that $B' := A[[y^{\pm 1};\alpha]]$ is semiprime, by e.g. [4, 10.18]. To start, let $f =$

$ay^i + a_{i+1}y^{i+1} + \cdots$ be a nonzero series in B', with initial nonzero term of degree $i \in \mathbb{Z}$, and with initial coefficient a. Suppose $fB'f = 0$. Then $f(y^{-i}A)f = 0$, and so $aAa = 0$, a contradiction since A is semiprime. Thus $fB'f \neq 0$ for all nonzero $f \in B'$, and so B' is semiprime.

Conversely, assume that B is semiprime. Since the zero ideal of B is α-stable, it follows from (2.6) that the zero ideal of B is a finite intersection of α-prime ideals. Therefore, by (i), the zero ideal of A is a finite intersection of α-prime ideals, and so A is semiprime, again by (2.6).

(iii) First of all, it is a straightforward exercise, using the Joseph-Small-Borho-Warfield Additivity Principle, to prove that the Goldie rank of a semiprime right (or left) noetherian ring cannot be less than the Goldie rank of any of its semiprime right (or left) noetherian subrings; see, e.g., [21, Theorem 1].

Now let E be the (right or left) Goldie quotient ring of A; then E is a semisimple artinian ring. Of course, $\operatorname{rank} A = \operatorname{rank} E$. Also, α extends to E, and B embeds as a ring into $E[[y;\alpha]]$. By (ii), $E[[y;\alpha]]$ is semiprime. It now follows from the preceding paragraph that

$$\operatorname{rank} A \leq \operatorname{rank} B \leq \operatorname{rank} E[[y;\alpha]].$$

To prove (iii), we can therefore assume without loss of generality that $A = E$.

Now set

$$A = V_1 \oplus \cdots \oplus V_d,$$

for simple right ideals $V_1, \ldots, V_d$ of A, with $d = \operatorname{rank} A$. Then

$$B = V_1B \oplus V_2B \oplus \cdots \oplus V_dB.$$

Therefore, it now suffices to prove that VB is uniform as a right B-module if V is an arbitrary simple right ideal of A.

Recall the $\langle y\rangle$-adic filtration of B described in (2.4i), and the associated graded ring $\operatorname{gr} B \cong A[z;\alpha]$. In particular, as noted in (2.4i), the $\langle y\rangle$-adic filtration on B is complete and $\operatorname{gr} B$ is noetherian. Thus, by (2.3ii), B is right zariskian with respect to the $\langle y\rangle$-adic filtration. It then follows from (2.3ii) that the right ideal VB of B is closed in the $\langle y\rangle$-adic topology. Since B is complete in the $\langle y\rangle$-adic topology, we see that VB is complete in the $\langle y\rangle$-adic topology on B.

Now consider the $\langle y\rangle$-adic filtration of VB,

$$VB = VB\langle y\rangle^0 \supseteq VB\langle y\rangle^1 \supseteq \cdots,$$

as in (2.3i). Note that VB, with this filtration, is a filtered right B-submodule of B as in (2.2iv). We further have the associated graded $\operatorname{gr} B$-module $U := \operatorname{gr}(VB)$. Since

$$\bigcap_i \langle y\rangle^i = 0,$$

it follows that

$$\bigcap_i VB\langle y\rangle^i \subseteq \bigcap_i \langle y\rangle^i = 0.$$

In other words, the filtration is separated. It is immediate that the filtration is exhaustive. Also, it follows from the preceding paragraph that the filtration is complete.

Since the $\langle y\rangle$-adic filtration of VB is exhaustive, separated, and complete, and since

$$U \cong V \otimes_A A[z;\alpha]$$

is a noetherian $\mathrm{gr}(B)$-module, it follows from (2.2vi) that the map $K \mapsto \mathrm{gr}\, K$, from right B-submodules K of VB to right $\mathrm{gr}\, B$-submodules $\mathrm{gr}\, K$ of U, preserves strict inclusions.

We now make two claims. The first is that every proper right B-module factor of VB has finite length as a right B-module, and the second is that VB itself does not have finite length as a right B-module. In view of the preceding paragraph, to prove the first claim we will instead prove that every proper $\mathrm{gr}\, B$-module factor of U has finite length as a right $\mathrm{gr}\, B$-module. To start, identify $\mathrm{gr}\, B$ with $A[z;\alpha]$ and U with $V \otimes_A A[z;\alpha]$. Let $g = v_0 \otimes 1 + v_1 \otimes z + \cdots + v_j \otimes z^j$ be a nonzero element of U, for $v_0, \ldots, v_j \in V$, and with $v_j \neq 0$. Set $v = v_j$. Then $v.A = V$, and so $v\otimes 1, v\otimes z, v\otimes z^2, \ldots$ generate U as a right A-module (noting that $v\otimes z^i.a = v\alpha^i(a)\otimes z^i$ for $a \in A$). It is not hard to see, now, that the images of $v\otimes 1, v\otimes z, v\otimes z^2, \ldots, v\otimes z^{j-1}$ generate $U/(g.A[z;\alpha])$ as a right A-module. Thus $U/(g.A[z;\alpha])$ has finite length over (the finite length algebra) A and so has finite length over $A[z;\alpha]$. The first claim follows.

To prove the second claim, assume that VB has finite length as a right B-module. Then $VB.(J(B))^\ell = 0$ for some positive integer ℓ, and as noted above, $J(B) = \langle y\rangle$. However, V contains at least one nonzero element v, and so $v.y^\ell \neq 0$, because y is a regular element of $B = A[[y;\alpha]]$. We have arrived at a contradiction, and the second claim follows: VB has infinite length as a right B-module.

We can now prove as follows that VB is a uniform right ideal of B. Suppose that W_1 and W_2 are nonzero B-submodules of VB such that $W_1 \cap W_2 = 0$. Then the infinite length right B-module VB embeds as a right B-module into the finite length right B-module $VB/W_1 \oplus VB/W_2$, a contradiction. Hence VB is uniform.

(iv) Statements (i) and (ii) follow for B' via [4, 10.18]. Statement (iii) follows since B' is an Ore localization of B.

The theorem follows. □

Combining (2.1) with (2.9), we obtain:

2.10. **Corollary.** *Assume that A is noetherian, and recall $B = A[[y;\alpha]]$. Let I be an α-ideal of A.* (i) *I is α-prime if and only if IB is α-prime.* (ii) *I is semiprime if and only if IB is semiprime.* (iii) *If I is semiprime then* $\mathrm{rank}\, B/IB = \mathrm{rank}\, A/I$. (iv) *Statements* (i) *through* (iii) *remain true if B is replaced by $B' = A[[y^{\pm 1};\alpha]]$.*

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