Comprehensive Examination in Algebra
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Part I. Do three of these problems.

I.1 Let G be a group (not necessarily finite). Prove:

(a) If $C, D \leq G$ are two subgroups of finite index in G, then the index $|G : C \cap D|$ is finite as well.

(b) If $C \leq G$ is a subgroup of finite index in G, then there is a normal subgroup $N \trianglelefteq G$ such that $N \subseteq C$ and G/N is finite.

I.2 Let A and B be $n \times n$ matrices with entries in an algebraically closed field F satisfying $AB = BA$. Show:

(a) A and B have a common eigenvector.

(b) Conclude by induction on n that there is an invertible $n \times n$ matrix C over F such that $C^{-1}AC$ and $C^{-1}BC$ are both upper triangular.

I.3 Let R be a commutative ring and let $I = (r_1, \ldots, r_s)$ be a finitely generated ideal of R.

(a) Show that all powers I^k $(k \geq 1)$ are finitely generated.

(b) If R/I is finite, show that all R/I^k $(k \geq 1)$ are finite.

I.4 For a given prime $p \in \mathbb{Z}$, let $^-\colon \mathbb{Z} \twoheadrightarrow \mathbb{F}_p = \mathbb{Z}/(p)$ denote the canonical epimorphism that is given by reduction modulo p. Consider the ring homomorphism $^-\colon \mathbb{Z}[x] \twoheadrightarrow \mathbb{F}_p[x]$ sending $a_n x^n + \cdots + a_0 \in \mathbb{Z}[x]$ to $\overline{a_n} x^n + \cdots + \overline{a_0} \in \mathbb{F}_p[x]$.

Let $f \in \mathbb{Z}[x]$ be a monic polynomial such that $\overline{f} \in \mathbb{F}_p[x]$ is irreducible.

(a) Show that $f(x)$ is irreducible over $\mathbb{Q}$.

(b) Let α be a root of f and $\overline{\alpha}$ a root of $\overline{f}$ (in some algebraic closures of $\mathbb{Q}$ and $\mathbb{F}_p$, resp.). Show that there is a ring epimorphism $\mathbb{Z}[\alpha] \twoheadrightarrow \mathbb{F}_p(\overline{\alpha})$ such that $p \mapsto 0$ and $\alpha \mapsto \overline{\alpha}$.

Part II. Do two of these problems.

II.1 Let G be a group (not necessarily finite) having a subnormal series

$$\langle 1 \rangle = G_0 \trianglelefteq G_1 \trianglelefteq \cdots \trianglelefteq G_r = G$$

such that G_i/G_{i-1} is cyclic for all $1 \leq i \leq r$. Prove:

(a) G can be generated by at most r elements.

(b) If H is a subgroup or a homomorphic image of G, then H has a subnormal series

$$\langle 1 \rangle = H_0 \trianglelefteq H_1 \trianglelefteq \cdots \trianglelefteq H_r = H$$

such that H_i/H_{i-1} is cyclic for all $1 \leq i \leq r$.

II.2 Consider the following subring of $M_2(\mathbb{Z})$ (you do not have to prove that it is indeed a subring):

$$R := \begin{bmatrix} \mathbb{Z} & \mathbb{Z} \\ 0 & \mathbb{Z} \end{bmatrix} = \left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \middle| \; a, b, c \in \mathbb{Z} \right\}.$$

Give a complete description of the maximal ideals of R.

II.3 Let F/K be a finite Galois extension and let $p \in \mathbb{Z}$ be a fixed prime number. Show that there exists a unique smallest intermediate field $E = E(p)$ such that (i) E/K is Galois and (ii) the degree $[F : E]$ is a power of p. (Here, "smallest" means that E is contained in all other intermediate fields having properties (i) and (ii).)