

Comprehensive Examination in Algebra
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PART I: Do three of the following problems.

1. An abelian group is *uniform* if every intersection of two nontrivial subgroups is also nontrivial.
 - (a) Let A be a nontrivial finite uniform abelian group. Prove that $A \cong \mathbb{Z}/\mathbb{Z}_p^n$ for some positive integer n and some prime number p .
 - (b) Suppose that A is a finitely generated infinite abelian group. Prove that A is uniform if and only if A is cyclic.
 - (c) Give a complete description, up to isomorphism, of the finitely generated uniform abelian groups.

2. Let $F = \mathbb{F}_3$ be the field with 3 elements and let $G = \text{GL}_2(F)/F^*$ denote the group of invertible 2×2 -matrices over F modulo the scalar matrices.
 - (a) Show that $|G| = 24$.
 - (b) Show that G acts on the set of all 1-dimensional subspaces of the vector space $V = F^2$, and only the identity element of G fixes all 1-dimensional subspaces.
 - (c) Conclude that G is isomorphic to the symmetric group S_4 .

3. Set $R = \mathbb{Z}[x]$, the polynomial ring in the single variable x , with integer coefficients. Prove that R is not a principal ideal domain.

4. Let F be a field and let α, β be distinct elements of F . Let $F[x]$ denote the polynomial algebra over F and define $R = \{f/g \mid f, g \in F[x], g(\alpha)g(\beta) \neq 0\}$.
 - (a) Show that R is a subring of $F(x)$, the field of rational functions over F .
 - (b) Determine the units of R .
 - (c) Determine the maximal ideals of R .

Part II: Do two of the following problems.

1. Let G be a group of order 231.
 - (a) Prove that G contains a unique Sylow 11-subgroup P .
 - (b) Determine $\text{Aut}(P)$, the group of automorphisms of P .
 - (c) Prove that there is a group homomorphism from G into $\text{Aut}(P)$.
 - (d) Prove that P is contained in the center of G .

2. Let n be a positive integer, and let k be an algebraically closed field of characteristic 0. Let X and Y be $n \times n$ matrices over k such that $XY - YX = Y$. Prove that X and Y have a common eigenvector. (Hint: First prove, if $v \in k^n$ is an eigenvector for X , that v, Yv, Y^2v, \dots span a vector subspace of k^n invariant under both X and Y .)

3. Let K/F be a finite Galois extension of fields, with Galois group $\Gamma = \text{Gal}(K/F)$. For $\alpha \in K$, define $\text{Tr}_{K/F}(\alpha) = \sum_{\gamma \in \Gamma} \gamma(\alpha)$.
 - (a) Prove that $\text{Tr}_{K/F}(\alpha) \in F$.
 - (b) Let $x^m + cx^{m-1} + \dots$ be the minimal polynomial of α over F . Show that m divides the degree $[K : F]$ and that $\text{Tr}_{K/F}(\alpha) = -\frac{[K:F]}{m} \cdot c$.