

**PH.D. COMPREHENSIVE EXAMINATION**  
**ABSTRACT ALGEBRA SECTION**

**January 1996**

**Part I.** Do three (3) of these problems.

**I.1.** a) Let  $G$  be a group with a subgroup  $H$  and a normal subgroup  $N$ . Prove that  $H \cap N$  is a normal subgroup of  $H$  and that  $H/(H \cap N) \cong HN/N$ .

b) Use the result in part (a) to prove that a subgroup  $H$  of a solvable group is solvable.  
(Note: A group  $G$  is solvable if there exists a subnormal series  $\{1\} = N_k \triangleleft N_{k-1} \triangleleft \cdots \triangleleft N_2 \triangleleft N_1 \triangleleft N_0 \triangleleft G$ , where, for each  $i = 1, \dots, k$ ,  $N_{i-1}/N_i$  is abelian.)

**I.2.** Let  $R$  be an Euclidean domain with unity.

a) Prove that  $R$  is a principal ideal domain.

For  $r$  and  $s$  in  $R$ , let  $\langle r, s \rangle = \{rm + sn \mid m, n \in R\}$

b) Prove that  $\langle r, s \rangle$  is an ideal of  $R$ .

c) Suppose that  $R = Q[x]$  (where  $Q$  represents the field of rational numbers), and that  $r = x^2 - 3x + 2$  and  $s = x^3 - 9x^2 + 23x - 15$ . Then, by parts (a) and (b),  $\langle r, s \rangle = \langle p \rangle$ , for some  $p \in Q[X]$ . Find  $p$ . (Show all work and justify your answer.)

**I.3.** Let  $F$  be a field. Then  $F[x]$  is a commutative ring with unity. (You may accept this without proof.)

a) Show that  $F[x]$  is an integral domain but that it cannot be a field.

Let  $E$  be an extension field of  $F$ . For  $\alpha \in E$ , let  $\phi : F[x] \rightarrow E$  be a homomorphism which fixes the elements of  $F$  and which maps  $x$  into  $\alpha$ .

b) Describe the kernel of  $\phi$ .

c) Prove that if  $\alpha$  is algebraic then the image of  $\phi$  is a subfield of  $E$ .

d) For  $F = Q$  (the rationals) and for  $\alpha = \sqrt[3]{2}$ , describe the image of  $\phi$  as a subfield of  $R$  (the reals). (I.e., find a basis and a unique representation for each element.)

**I.4.** Let  $V$  be a finite dimensional vector space with an inner product  $\langle \cdot, \cdot \rangle$ . Fix a non-zero vector  $u$  in  $V$  and define a mapping  $T : V \rightarrow V$  by  $T(v) = \langle u, v \rangle u$ .

a) Show that  $T$  is a linear transformation.

b) Find the characteristic polynomial, eigenvalues, minimal polynomial, and Jordan canonical form of  $T$ .

c) If, instead of  $T$ , we consider the mapping  $F : V \rightarrow V$  given by  $F(v) = \langle u, v \rangle v$ , explain why the analogues of the questions asked in part b) are not meaningful for  $F$ .

**Part II.** Do two (2) of these problems.

**II.1.** a) Find all groups of order 325.

b) Find all groups of order 22.

**II.2.** Let  $A$  be an  $m \times n$  matrix over a field  $F$ .

a) Show that the rank of  $A$  is equal to the smallest integer  $r$  such that  $A$  can be factored as  $A = BC$  for suitable matrices  $B$  and  $C$  of sizes  $m \times r$  and  $r \times n$  respectively.

b) Use part (a) to deduce the familiar fact that “row rank = column rank”.

**II.3.** Let  $F$  be a finite field with  $q$  elements.

a) Prove that the product of the non-zero elements of  $F$  is  $-1$ .

b) Prove that if  $q$  is even then every element of  $F$  is a square; and that if  $q$  is odd, then the set of non-zero squares of  $F$  is a subgroup of index 2 of the group of non-zero elements of  $F$ .