

August, 1998

Comprehensive Examination

Department of Mathematics

ALGEBRA

PART I: Do three of the following problems.

1. Let  $G$  be a group.
  - (a) Show that  $G$  is finite iff  $G$  has only finitely many distinct subgroups.
  - (b) Show that  $G$  has exactly 3 distinct subgroups iff  $G$  is cyclic of order  $p^2$  for some prime  $p$ .
2. Let  $A$  be a real  $r \times r$ -matrix satisfying  $A^n = I$  for some  $n > 0$ . Prove:  $\det A = (-1)^m$ , where  $m$  is the multiplicity of  $-1$  as root of the characteristic polynomial of  $A$ .
3. Let  $R$  be a ring and let  $N$  be an ideal of  $R$  such that every element of  $x \in N$  is nilpotent, that is,  $x^t = 0$  for some  $t$ . Show that, under the canonical map  $R \rightarrow R/N$ , the group of units  $U(R)$  of  $R$  maps *onto* the group of units  $U(R/N)$  of  $R/N$ . (Recall that a *unit* of a ring  $R$  is an invertible element of  $R$ .)
4. Let  $F \supseteq K$  be an algebraic extension of fields and let  $R$  be a subring of  $F$  with  $R \supseteq K$ . Show that  $R$  is a field.

PART II: Do two of the following problems.

1. Let  $G$  be a group of order  $pqr$  with distinct primes  $p$ ,  $q$ , and  $r$ . Show that  $G$  is not simple.
2. Let  $R = K[x, y]$  be the ring of polynomials in two variables  $x$  and  $y$  with coefficients in the field  $K$ , and let  $f(x, y) \in R$ .
  - (a) Show that the principal ideal of  $R$  that is generated by  $f(x, y)$  is prime if and only if the polynomial  $f(x, y)$  is irreducible.
  - (b) Show that the ideal of  $R$  that is generated by  $x$  and  $f(x, y)$  is maximal if and only if the polynomial  $f(0, y)$  is irreducible in  $K[y]$ .
3. Let  $F$  be the splitting field of  $x^6 - 3$  over  $\mathbf{Q}$ .
  - (a) Show that  $[F : \mathbf{Q}] = 12$ .
  - (b) Let  $G = \text{Gal}(F/\mathbf{Q})$ . Show that there exist a normal subgroup  $H$  of  $G$  of order 6 and a subgroup  $K$  of  $G$  of order 2 such that  $G$  is a semidirect product of  $H$  and  $K$ .
  - (c) Determine whether the subgroup  $H$  of Part (b) is abelian.