PhD Algebra Exam
Fall 1994

Part I: Do three of these problems.

1. Find all the integers which are orders of elements of the alternating group A_5. Count how many elements of A_5 are of each of those orders and describe all the elements of each order.

2. Give short explanations:

(1) Why is a field necessarily a p. i. d.?
(2) Why is every subgroup of a solvable group solvable?
(3) Why is a field extension of finite degree necessarily algebraic?
(4) Why is the number of elements of a finite field necessarily a prime power?
(5) Why is an abelian simple group necessarily of prime order?

3. Suppose that V is a finite dimensional vector space, T a nilpotent linear transformation on V. Let $n = \dim V$. Show that $\dim T^k(V) \leq n - k$, and thus $T^n = 0$. What does it say about the Jordan normal form of T if all the inequalities are equality? Give a non-trivial example of a T for which some of the inequalities are not equality.

4. Consider the rings $A = \mathbb{Q}[x]/(x^2 - 2x)$, $B = \mathbb{Q}[x]/(x^2 - 1)$, $C = \mathbb{Q}[x]/(x^2)$. ($\mathbb{Q}$ = rational numbers.) Show that A and B are isomorphic, but B and C are not isomorphic.

Part II: Do two of these problems.

5. Let F be the finite field with p elements (p is a prime), and let $GL(n, F)$ be the group of invertible $n \times n$ matrices with entries in F.

(1) Determine the order of $GL(2, F)$.
(2) Find a p-Sylow subgroup of $GL(2, F)$.
(3) Determine the order of $GL(3, F)$.

6. Show that the only group of order 8 which is isomorphic to a subgroup of the symmetric group S_4 is the dihedral group D_4.

7. Determine the structure of $\mathbb{Z}_{15}^{\times}$, the group of units of $\mathbb{Z}_{15}$ (= integers mod 15). Let K be the cyclotomic field $\mathbb{Q}(z)$, where z is a primitive 15^{th} root of unity. Exhibit an isomorphism of the Galois group $G(K/\mathbb{Q})$ with $\mathbb{Z}_{15}^{\times}$.