ON THE PARAMETERIZATION OF PRIMITIVE IDEALS IN AFFINE PI ALGEBRAS

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ABSTRACT. We consider the following question, concerning associative algebras R over an algebraically closed field k: When can the space of (equivalence classes of) finite dimensional irreducible representations of R be topologically embedded into a classical affine space? We provide an affirmative answer for algebraic quantum groups at roots of unity. More generally, we give an affirmative answer for k-affine maximal orders satisfying a polynomial identity, when k has characteristic zero. Our approach closely follows the foundational studies by Artin and Procesi on finite dimensional representations. Our results also depend on Procesi's later study of Cayley-Hamilton identities.

1. INTRODUCTION

1.1. Let k be an algebraically closed field, and let R be an associative k-algebra with generators $X_1, \dots, X_s$. In the foundational studies of Artin [1], in 1969, and Procesi [9], in 1974, it was shown that the semisimple n-dimensional representations of R (over k) were parametrized up to equivalence by a closed subset of $\operatorname{Max} \mathbb{T}(n,s)$, where $\mathbb{T}(n,s)$ is the affine (i.e., finitely generated) commutative k-algebra generated by the coefficients of the characteristic polynomials of s-many generic $n{\times}n$ matrices. It was further shown by Artin and Procesi in [1; 9] that $\operatorname{Prim}_n R$, the set of kernels of n-dimensional irreducible representations of R, is homeomorphic to a locally closed subset of $\operatorname{Max} \mathbb{T}(n,s)$. (Here and throughout, the Jacobson/Zariski topology is employed.) In particular, when the irreducible representations of R all have dimension n (e.g., when R is an Azumaya algebra of rank n, by what is now known as the Artin-Procesi theorem [1; 9]), the space $\operatorname{Prim} R$ of kernels of irreducible representations of R is homeomorphic to a locally closed subset of affine space. In this note we examine generalizations of this embedding for more general classes of k-affine PI (i.e., polynomial identity) algebras. Our analysis closely follows the above cited work of Artin and Procesi, and also depends on the later study by Procesi of Cayley-Hamilton identities [7].

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1.2. Our main result, proved in (5.4):

Theorem. *Let A be a prime affine PI algebra over an algebraically closed field k of characteristic zero, and suppose that A is a maximal right (or left) order in a simple artinian ring Q. Then* $\operatorname{Prim} A$ *is homeomorphic to a constructible subset of the affine space k^N, for a suitable choice of positive integer N.*

Examples to which the theorem applies include algebraic quantum groups at roots of unity. Recent studies of quantum groups from this general point of view include [4].

1.3. For an arbitrary k-affine PI algebra R, in arbitrary characteristic, we are able to construct a closed bijection from $\operatorname{Prim} R$ onto a constructible subset of k^N, again for a suitable choice of N. We therefore ask whether the conclusion of the preceding theorem holds for all k-affine PI algebras.

1.4. We assume that the reader is familiar with the basic theory of PI algebras; general references include [6, Chapter 13], [8], and [11].

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2. CONSTRUCTING THE INJECTION Ψ

Our goal in this section is to construct an injection, specified in (2.11), from $\operatorname{Prim} R$ into the maximal spectrum of a suitable "trace ring." The approach is directly adapted from [9], with some added bookkeeping.

2.1 (R, d, N) First Notation and Conventions. The following will remain in effect throughout this paper.

(i) Set

$$R = k\left\{\widehat{X}_1, \widehat{X}_2, \dots, \widehat{X}_s\right\} \Big/ \left\langle \widehat{f}_1, \widehat{f}_2, \dots \right\rangle,$$

the factor of the free associative k-algebra in the generators $\widehat{X}_1, \widehat{X}_2, \dots, \widehat{X}_s$ modulo the (not necessarily finitely many) relations $\widehat{f}_1, \widehat{f}_2, \dots$. Let X_ℓ denote the canonical image of $\widehat{X}_\ell$ in R, for each ℓ. Assume that R satisfies a (monic) polynomial identity.

(ii) All k-algebra homomorphims mentioned will be assumed to be unital. A *representation* is a k-algebra homomorphism into the algebra of linear operators on a k-vector space. If Γ is a k-algebra we will assume that the sets $\operatorname{Prim}\Gamma$ of (left) primitive ideals and $\operatorname{Max}\Gamma$ of maximal ideals are equipped with the Jacobson/Zariski topology: The closed sets have the form $V_\Gamma(I) = \{P : P \supseteq I\}$ for ideals I of Γ.

(iii) Recall from Kaplansky's Theorem and standard PI theory that there exists a positive integer d such that every irreducible representation of R has (k-)dimension no greater than d. Let N be a common multiple of $1, 2, \dots, d$. (Note that our choices of d and N remain valid when R is replaced with a homomorphic image.)

(iv) Repeatedly-used non-standard notation will be listed (within parentheses) at the beginning of the subsection in which it is introduced.

2.2 ($\widehat{C}_n$, $\widehat{x}_{ij}^{(\ell,n)}$, $\mathbb{G}(n,s)$, $\mathbb{T}(n,s)$, $\widehat{M}_n$). Let n be a positive integer. Set

$$\widehat{C}_n = k\left[\ \widehat{x}_{ij}^{(\ell,n)}\ :\ 1 \le i,j \le n,\ \ell = 1,2,\dots\ \right],$$

the commutative polynomial k-algebra in the variables $\widehat{x}_{ij}^{(\ell,n)}$. Also, set $\widehat{M}_n = M_n(\widehat{C}_n)$, the k-algebra of $n{\times}n$ matrices with entries in $\widehat{C}_n$. Identify $\widehat{C}_n$ with the $\widehat{C}_n$-scalar matrices in $\widehat{M}_n$; in other words, identify $\widehat{C}_n$ with the center $Z(\widehat{M}_n)$ of $\widehat{M}_n$.

Let $\mathbb{G}(n,s)$ denote the k-subalgebra of $\widehat{M}_n$ generated by the generic matrices

$$\left(\widehat{x}_{ij}^{(1,n)}\right)_{n\times n}, \dots, \left(\widehat{x}_{ij}^{(s,n)}\right)_{n\times n}.$$

The k-subalgebra of $\widehat{C}_n$ generated by the coefficients of the characteristic polynomials of the elements of $\mathbb{G}(n,s)$ will be denoted $\mathbb{T}(n,s)$. It is a well known consequence of Shirshov's Theorem that $\mathbb{T}(n,s)$ is k-affine [9, 3.1]. It is also well known that $\mathbb{T}(n,s)$ is generated, in characteristic zero, by the traces of the elements of $\mathbb{G}(n,s)$.

2.3 (Rel($\widehat{C}_n$), Rel($\widehat{M}_n$)). Now consider the k-algebra homomorphism

$$k\left\{\widehat{X}_1,\dots,\widehat{X}_s\right\} \xrightarrow{\ \widehat{\pi}_n\ } \widehat{M}_n,$$

mapping

$$\widehat{X}_\ell \longmapsto \left(\widehat{x}_{ij}^{(\ell,n)}\right)_{n\times n},$$

for each ℓ. Let Rel($\widehat{C}_n$) be the ideal of $\widehat{C}_n$ generated by the entries of

$$\widehat{\pi}_n\left(\widehat{f}_1\right),\ \widehat{\pi}_n\left(\widehat{f}_2\right),\ \dots,$$

and let Rel($\widehat{M}_n$) be the ideal of $\widehat{M}_n$ generated by Rel($\widehat{C}_n$). Then

$$\operatorname{Rel}\left(\widehat{M}_n\right) = M_n\left(\operatorname{Rel}\left(\widehat{C}_n\right)\right), \quad \text{and} \quad \operatorname{Rel}\left(\widehat{C}_n\right) = \operatorname{Rel}\left(\widehat{M}_n\right) \cap \widehat{C}_n.$$

2.4 (C_n, M_n, $x_{ij}^{(\ell,n)}$, π_n, T_n). Set

$$C_n \ = \ \widehat{C}_n\big/\operatorname{Rel}\left(\widehat{C}_n\right), \quad \text{and} \quad M_n \ = \ M_n\left(C_n\right) \ \cong \ \widehat{M}_n\big/\operatorname{Rel}\left(\widehat{M}_n\right).$$

Denote the natural image of each $\widehat{x}_{ij}^{(\ell,n)}$ in C_n by $x_{ij}^{(\ell,n)}$. We obtain a k-algebra homomorphism

$$\pi_n\colon R \xrightarrow{\ X_\ell \longmapsto \left(x_{ij}^{(\ell,n)}\right)_{n\times n}\ } M_n.$$

Note that $\pi_n(R)$ is a natural image of $\mathbb{G}(n,s)$.

Identify C_n with $Z(M_n)$, and let $T_n = T_n(R)$ denote the k-subalgebra of C_n generated by the coefficients of the characteristic polynomials of the elements of $\pi_n(R)$. Observe that T_n is a natural image of $\mathbb{T}(n,s)$.

2.5. Say that a k-algebra homomorphism $h\colon M_n \to M_n(k)$ is *matrix unital* if h restricts to the identity map on $M_n(k) \subseteq M_n$. Letting e_{ij} denote the ijth matrix unit of $M_n(k)$, we see that h is matrix unital if and only if $h(e_{ij}) = e_{ij}$ for all i and j.

2.6 ($\widetilde{\rho}$). Now Let $\rho\colon R \to M_n(k)$ be a representation. Observe that there is a unique matrix unital k-algebra homomorphism $\widetilde{\rho}\colon M_n \to M_n(k)$ such that the following diagram commutes:

$$\begin{tikzcd}
R \arrow[r, "\pi_n"] \arrow[d, equal] & M_n \arrow[d, "\widetilde{\rho}"] & C_n \arrow[l, hook', "\text{inclusion}"'] \arrow[d, "\widetilde{\rho}|_{C_n}"] & T_n \arrow[l, hook', "\text{inclusion}"'] \arrow[d, "\widetilde{\rho}|_{T_n}"] \\
R \arrow[r, "\rho"] & M_n(k) & k \arrow[l, hook', "\text{inclusion}"'] \arrow[r, equal] & k
\end{tikzcd}$$

Of course, every k-algebra homomorphism $C_n \to k$ produces a representation $R \to M_n(k)$ in an obvious way.

2.7 (Θ_n). Let $\operatorname{Rep}_n R$ denote the set of n-dimensional representations of R (without identifying equivalence classes), and let $\operatorname{Alg}(T_n, k)$ denote the set of k-algebra homomorphisms from T_n onto k. We have a function

$$\Theta_n\colon \operatorname{Rep}_n(R,k) \xrightarrow{\rho \longmapsto \widetilde{\rho}|_{T_n}} \operatorname{Alg}(T_n,k) \cong \operatorname{Max} T_n.$$

For a given representation $\rho\colon R \to M_n(k)$, let semisimple(ρ) denote the unique equivalence class of semisimple n-dimensional representations corresponding to ρ (i.e., the semisimple representations obtained from the direct sum of the Jordan-Hölder factors of the R-module associated to ρ). We now recall:

Theorem. (Artin [1, §12]; Procesi [9, §4]) (a) Θ_n *is surjective.* (b) $\Theta_n(\rho) = \Theta_n(\rho')$ *if and only if* semisimple(ρ) = semisimple(ρ').

2.8 (γ_P, Φ_m). (i) Let $\operatorname{Prim}_m R$ denote the set of (left) primitive ideals of rank m (i.e., the set of kernels of m-dimensional irreducible representations of R). Note that $1 \leq m \leq d$. Equip $\operatorname{Prim}_m R$ with the relative topology, viewing it as a subspace of $\operatorname{Prim} R$. As noted in [1, §12] and [9, §5], $\operatorname{Prim}_m R$ is a locally closed subset of $\operatorname{Prim} R$.

(ii) Choose $P \in \operatorname{Prim}_m R$. Then P uniquely determines an equivalence class of irreducible m-dimensional representations; choose $\rho\colon R \to M_m(k)$ in this equivalence class. Let γ_P denote the k-algebra homomorphism $\widetilde{\rho}|_{T_m}\colon T_m \to k$. By (2.7), γ_P depends only on P, and we obtain an injection

$$\Phi_m\colon \operatorname{Prim}_m R \xrightarrow{P \,\longmapsto\, \ker\gamma_P} \operatorname{Max} T_m.$$

(iii) It follows from [1, §12] and [9, §5] that the image of Φ_m is an open subset of $\operatorname{Max} T_m$ and that Φ_m is homeomorphic onto its image.

2.9 (ρ_N). Now choose a positive integer m no greater than d, and let $\rho\colon R \to M_m(k)$ be a representation. We will use $\rho_N\colon R \to M_N(k)$ to denote the associated N-dimensional diagonal representation, mapping

$$r \longmapsto \begin{bmatrix} \rho(r) & & & \\ & \rho(r) & & \\ & & \ddots & \\ & & & \rho(r) \end{bmatrix},$$

for $r \in R$.

2.10 (C, π, M, T, $x_{ij}^{(\ell)}$). In the remainder of this note we mostly will be concerned with the case when $n = N$, and so we will set $C = C_N$, $\pi = \pi_N$, $M = M_N$, $T = T_N = T_N(R) = T(R)$, and

$$\left(x_{ij}^{(\ell)}\right) = \left(x_{ij}^{(\ell,N)}\right)_{n\times n}.$$

2.11 ($\gamma_{N,P}$, Ψ) The injection. Now let P be a primitive ideal of R. Proceeding as before, P uniquely determines an equivalence class of irreducible m-dimensional representations for some $1 \leq m \leq d$; choose $\rho\colon R \to M_m(k)$ in this equivalence class. Combining (2.6) and (2.9), let $\gamma_{N,P}$ denote the k-algebra homomorphism $(\widetilde{\rho_N})|_T\colon T \to k$. We can now define an injection:

$$\Psi\colon \operatorname{Prim} R \xrightarrow{P \longmapsto \ker \gamma_{N,P}} \operatorname{Max} T$$

In §3 the image of Ψ will be described. In §4 it will be proved that Ψ is open (and closed) onto its image. In §5 it will be seen, in certain special cases, that Ψ is homeomorphic onto its image.

Note now, however, that implicit in the preceding is a natural (and obvious) homeomorphism between $\operatorname{Prim} R$ and $\operatorname{Prim} \pi(R)$.

2.12. Choose P, m, and ρ as in (2.11). Up to equivalence, there is exactly one N-dimensional representation of R with kernel P, namely, the representation corresponding to the unique (up to isomorphism) semisimple R/P-module of length N/m. Therefore, by (2.7), $\gamma_{N,P}$ depends only on P and not our specific choice ρ_N of N-dimensional representation.

3. The image of Ψ

Retain the notation of the preceding section. Throughout this section, m will denote a positive integer no greater than d. The main result of this section, (3.7), explicitly determines the image of Ψ; in particular, the image is a constructible subset.

3.1. Given an $N\times N$ matrix, the (N/m)-many $m\times m$ blocks running consecutively down the main diagonal will form the *m-block diagonal*. An $N\times N$ matrix with only zero entries off the m-block diagonal will be referred to as an *m-block diagonal matrix*.

3.2. Consider the k-algebra homomorphism $\widehat{C} \to \widehat{C}_m$ mapping the ijth entry of $\left(\widehat{x}_{ij}^{(\ell)}\right) \in \widehat{M}$ to the ijth entry of the m-block diagonal matrix

$$\begin{bmatrix} \left(\widehat{x}_{ij}^{(\ell,m)}\right)_{m\times m} & & \\ & \ddots & \\ & & \left(\widehat{x}_{ij}^{(\ell,m)}\right)_{m\times m} \end{bmatrix} \in M_N\left(\widehat{C}_m\right).$$

We obtain a commutative diagram of k-algebra homomorphisms:

$$\begin{tikzcd}
\widehat{C} \arrow[r] \arrow[d, "\text{projection}"'] & \widehat{C}_m \arrow[d, "\text{projection}"] \\
C \arrow[r] & C_m \\
T \arrow[r] \arrow[u, "\text{inclusion}"] & T_m \arrow[u, "\text{inclusion}"']
\end{tikzcd}$$

We will refer to the horizontal maps as *specializations.*

3.3. Note that the preceding maps $\widehat{C} \to \widehat{C}_m$ and $C \to C_m$ are surjective. In characteristic zero, T and T_m are generated, respectively, by the traces of the matrices contained in $\pi(R)$ and $\pi_m(R)$, and it follows in this situation that the specialization $T \to T_m$ is surjective. In arbitrary characteristic, it is not hard to see that the specialization $T \to T_m$ is an integral ring homomorphism.

3.4 (H_m, I_m, J_m). Let H_m denote the kernel of the specializtion $C \to C_m$. In other words, H_m is the ideal of C generated by the sets

$$\left\{ x_{ij}^{(\ell)} \;\middle|\; \begin{array}{l} x_{ij}^{(\ell)} \text{ is not within the } m\text{-block diagonal of } \left(x_{ij}^{(\ell)}\right); \\ 1 \le i,j \le N;\ \ell = 1,2,\dots \end{array} \right\}$$

and

$$\left\{ x_{ij}^{(\ell)} - x_{i'j'}^{(\ell)} \;\middle|\; \begin{array}{l} x_{ij}^{(\ell)} \text{ and } x_{i'j'}^{(\ell)} \text{ are within the } m\text{-block diagonal of } \left(x_{ij}^{(\ell)}\right); \\ i = i' \pmod{m} \text{ and } j = j' \pmod{m};\ 1 \le i,j \le N;\ \ell = 1,2,\dots \end{array} \right\}.$$

Set $I_m = MH_m = M_N(H_m)$. Let $J_m = H_m \cap T = I_m \cap T$ denote the kernal of the specialization $T \to T_m$.

3.5. Now suppose that P is the kernel of an irreducible representation $\rho\colon R \to M_m(k)$. Recalling the notation of (2.6) and (2.9), we see that the kernel of $\widetilde{\rho}_N\colon M \to M_N(k)$ contains I_m, and so the kernel of $\widetilde{\rho}|_T$ contains J_m. We conclude that Ψ maps $\operatorname{Prim}_m R$ into the set $V_T(J_m)$ of maximal ideals of T containing J_m.

3.6 (E_m). We now proceed in a fashion similar to [9, §5], employing the central polynomials of Formanek [5] or Razmyslov [10]. To start (see, e.g., [6, §13.5] or [11, §1.4] for details), we can construct a polynomial p_m in noncommuting indeterminates with the following two properties, holding for all commutative rings Λ with identity: First, $p_m(M_m(\Lambda)) \subseteq Z(M_m(\Lambda))$, and second, $p_m(M_m(\Lambda))$ generates $M_m(\Lambda)$ as an additive group. (Here, $p_m(M_m(\Lambda))$ refers to all evaluations of p_m where the indeterminates have been substituted with elements of $M_m(\Lambda)$.) Hence, $\pi\big(p_m(R)\big) = p_m\big(\pi(R)\big)$ is contained, modulo I_m, within the center of M. Moreover, a representation $R \to M_m(k)$ is irreducible if and only if $p_m(R)$ is not contained in the kernel, if and only if the image of $p_m(R)$ generates as an additive group the full set of scalar matrices in $M_m(k)$.

Next, modulo I_m, the characteristic polynomial of $c \in \pi\big(p_m(R)\big)$ is $(\lambda - c)^N$, and so $c^N \in T + I_m$. We can therefore choose a set $E_m \subseteq T$ of transversals in M for

$$\{\ c^N + I_m \ :\ c \in \pi\big(p_m(R)\big)\ \},$$

with respect to I_m.

Now let $\varphi\colon R \to M_m(k)$ be a representation. As in (3.5), $I_m \subseteq \ker \widetilde{\varphi}_N$ and $J_m \subseteq \ker \widetilde{\varphi}|_T$. Observe that

$$\begin{aligned} \varphi \text{ is irreducible} \quad &\Longleftrightarrow \quad \varphi(p_m(R)) \neq 0 \quad \Longleftrightarrow \quad \widetilde{\varphi_N}\left(\pi\big(p_m(R)\big)\right) \neq 0 \\ &\Longleftrightarrow \quad \widetilde{\varphi_N}(E_m) \neq 0 \quad \Longleftrightarrow \quad \widetilde{\varphi_N}|_T(E_m) \neq 0. \end{aligned}$$

3.7. Let K_m be the ideal of T generated by E_m and J_m.

Theorem. (i) Ψ *maps* $\operatorname{Prim}_m R$ *onto* $V_T(J_m) \setminus V_T(K_m)$.

(ii) *The image of* Ψ *is*

$$\bigcup_{m=1}^{d} V_T(J_m) \setminus V_T(K_m).$$

In particular, the image of Ψ *is a constructible subset of* $\operatorname{Max} T$.

Proof. Immediate from (3.5) and (3.6). □

3.8. We ask: Can the image of Ψ be described in a simpler fashion? Is there a simple way to specify how the locally closed subsets in (3.7) fit together?

4. Ψ IS OPEN AND CLOSED ONTO ITS IMAGE

Retain the notation of §2 and §3. We now begin to consider the topological properties of Ψ. In (4.2) it is shown that Ψ is open and closed onto its image.

4.1. Let $R \to R'$ be a k-algebra homomorphism. As described in [9, pp. 177–178], the construction in (2.11) is functorial in the following sense.

(i) To start, we have a commutative diagram:

$$\begin{tikzcd} R \arrow[r, "\pi"] \arrow[d] & M \arrow[d] & T(R) \arrow[l] \arrow[d] \\ R' \arrow[r, "\pi'"] & M' & T(R') \arrow[l] \end{tikzcd}$$

Moreover, if $R \to R'$ is surjective then so too is $T(R) \to T(R')$.

(ii) Next, assuming that I is an ideal of R, that $R' = R/I$, and that $R \to R'$ is the natural projection, we obtain a commutative diagram:

$$\begin{tikzcd} \operatorname{Prim} R \arrow[r, "\Psi"] & \operatorname{Max} T \\ \operatorname{Prim} R' \arrow[r, "\Psi'"] \arrow[u, "P/I \longmapsto P"] & \operatorname{Max} T' \arrow[u, "\frac{\mathfrak{m}}{\left(M\pi(I)M\right)\cap T} \longmapsto \mathfrak{m}"'] \end{tikzcd}$$

Each arrow represents an injection, and each vertical arrow represents a topological embedding onto a closed subset.

The kernel of the homomorphism $T \to T'$ is $\big(M\pi(I)M\big) \cap T$.

4.2. Now let I be an arbitrary ideal of R, with corresponding closed subset $V_R(I)$ of $\operatorname{Prim} R$. Set

$$J = \big(M\pi(I)M\big) \cap T.$$

Proposition. *Ψ is open and closed onto its image. In particular,*

$$\Psi\big(V_R(I)\big) \;=\; \operatorname{Image}\Psi \cap \big(V_T(J)\big), \quad \textit{and} \quad \Psi\big(W_R(I)\big) \;=\; \operatorname{Image}\Psi \cap \big(W_T(J)\big),$$

where $W_R(I)$ denotes the complement of $V_R(I)$.

Proof. This follows from (4.1ii) and the injectivity of Ψ. $\square$

4.3. Combining (2.8) and (3.2), we have a commutative diagram:

$$\begin{tikzcd}
\operatorname{Prim} R \arrow[r, "\Psi"] & \operatorname{Max} T \\
\operatorname{Prim}_m R \arrow[u, "\text{inclusion}"] \arrow[r, "\Phi_m"] & \operatorname{Max} T_m \arrow[u, "\mathfrak{m} \longmapsto \text{specializtion}^{-1}(\mathfrak{m})"']
\end{tikzcd}$$

We conclude from (2.8iii) that the restriction of Ψ to $\operatorname{Prim}_m R$ is continuous. Recalling from (2.8iii) that $\operatorname{Prim}_m R$ is a locally closed subset of $\operatorname{Prim} R$, we may view this last conclusion as an assertion that Ψ is "piecewise continuous." Also, we can conclude that the preimage under Ψ of a constructible subset of $\operatorname{Max} T$ is constructible.

4.4. We ask: Is Ψ necessarily continuous? More generally, is $\operatorname{Prim} R$ homeomorphic to a subspace of affine n-space, for sufficiently large n? A partial answer is given in §5.

5. Applications to Algebras Satisfying Cayley-Hamilton Identities

We retain the notation of the previous sections, but assume in this section that k has characteristic zero. Our approach closely follows [7].

5.1 [7, §2] **Formal Traces and Cayley-Hamilton Identities.** Let Γ be a k-algebra.

(i) [7, 2.3] Say that Γ is equipped with a *(formal) trace (over k)* provided there exists a k-linear function $\operatorname{tr}\colon \Gamma \to \Gamma$ such that for all $a, b \in \Gamma$,

$$\operatorname{tr}(a)b = b\operatorname{tr}(a), \quad \operatorname{tr}(ab) = \operatorname{tr}(ba), \quad \text{and} \quad \operatorname{tr}(\operatorname{tr}(a)b) = \operatorname{tr}(a)\operatorname{tr}(b).$$

(ii) [7, 2.4] Suppose that Γ is equipped with a trace tr. For each $r \in \Gamma$, set

$$\chi_r^{(n)}(t) \;=\; \prod_{i=1}^{n}(t - t_{r,i}),$$

where the $t_{r,i}$ are "formal eigenvectors" for r satisfying

$$\sum_{i=1}^{n} t_{r,i}^{j} \;=\; \operatorname{tr}(r^j),$$

for all non-negative integers j. Say that Γ satisfies the *n-th Cayley-Hamilton identity* if $\chi_r^{(n)}(r) = 0$ for all $r \in \Gamma$.

(iii) Suppose that Γ is equipped with a trace tr. By [7, Theorem], there exists a commutative k-algebra Λ, and a trace compatible k-algebra embedding of Γ into $M_n(\Lambda)$, if and only if Γ satsifies the n-th Cayley-Hamilton identity.

(iv) (Cf. [4, 3.10].) Let p be a positive integer, let Λ be a commutative k-algebra, and suppose that there is a trace compatible k-algebra embedding $\Gamma \to M_n(\Lambda)$. The block diagonal embedding of $M_n(\Lambda)$ into $M_{pn}(\Lambda)$ then provides a trace compatible embedding of Γ into $M_{pn}(\Lambda)$.

(v) Suppose that Γ is equipped with a trace. We conclude from (iii) and (iv) that if Γ satisfies the n-th Cayley-Hamilton identity then Γ satisfies the Cayley-Hamilton identity for all positive multiples of n.

5.2. Returning to the setting of the previous sections (but with k now having characteristic zero), suppose that R, as in (2.1), satisfies the N-th Cayley-Hamilton identity. It then follows directly from [7, 2.6] (the main theorem in [7]) that T is contained in $\pi(R)$, the image of $\pi\colon R \to M$. Since T must be central in $\pi(R)$, and since every irreducible representation of R is finite dimensional over k, it follows from well known arguments that the function

$$\operatorname{Prim}\pi(R) \xrightarrow{P \longmapsto P\cap T} \operatorname{Max} T$$

is continuous. Furthermore, as noted in (2.11), π produces a natural homeomorphism between $\operatorname{Prim} R$ and $\operatorname{Prim}\pi(R)$. However, the composition

$$\operatorname{Prim} R \xrightarrow{\text{natural homeomorphism}} \operatorname{Prim}\pi(R) \xrightarrow{P\mapsto P\cap T} \operatorname{Max} T$$

is precisely the function Ψ of (2.11), which must therefore be continuous.

We obtain:

5.3 Proposition. *(Recall that k has characteristic zero.) Suppose that R satisfies the N-th Cayley-Hamilton identity. Then $\Psi\colon \operatorname{Prim} R \to \operatorname{Max} T$ is homeomorphic onto its image.*

Proof. That Ψ is closed onto its image follows from (4.2). That Ψ is continuous onto its image follows from (5.2). □

Combining (5.3) with our previous analysis produces our main result:

5.4 Theorem. *Let A be a prime affine PI algebra over an algebraically closed field k of characteristic zero, and suppose that A is a maximal right (or left) order in a simple artinian ring Q. Further suppose that Q has rank d, that A is a maximal order in Q, and that N is a common multiple of $1, 2, \ldots, d$. Then $\operatorname{Prim} A$ is homeomorphic to a constructible subset of the affine space k^N.*

Proof. To start, the irreducible representations of A all have dimension no greater than d. Next, A is equipped with both a trace and a trace compatible embedding into $d\times d$ matrices over a commutative ring, since A is a maximal order in Q; see (e.g.) [6, §13.9]. Hence, by (5.1iii), A satisfies the d-th Cayley-Hamilton identity, and so, by (5.1iv), A satisfies the N-th Cayley-Hamilton identity. The theorem now follows from (3.7) and (5.3). □

5.5 Quantum groups. For suitable complex roots of unity ϵ, the quantum enveloping algebras U_ϵ and quantum function algebras F_ϵ are prime affine PI $\mathbb{C}$-algebras and are maximal orders; see [2] and (e.g.) [3]. In particular, (5.4) applies to these algebras.

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