

VECTOR BUNDLES

Throughout we fix naturals n and k . A set $U \subset \mathbf{R}^{n+k}$ is **vector bundle open** if it is of the form $U_1 \times \mathbf{R}^k$ for some U_1 open in \mathbf{R}^n . Let E be a set. A **vector bundle parametrization** of E is a pair (X, U) where $U \subset \mathbf{R}^{n+k}$ is vector-bundle open and $X : U \rightarrow E$ is a bijection. If (X, U) and (Y, V) are two vector-bundle parametrizations, we say they are **vector-bundle compatible** if $X^{-1}(X(U) \cap Y(V)) = U_1 \times \mathbf{R}^k$ and $Y^{-1}(X(U) \cap Y(V)) = V_1 \times \mathbf{R}^k$ are vector-bundle open and $Y^{-1} \circ X : U_1 \times \mathbf{R}^k \rightarrow V_1 \times \mathbf{R}^k$ is of the form

$$Y^{-1} \circ X(a, b) = (f(a), g(a)b), \quad a \in U_1, b \in \mathbf{R}^k,$$

for some differentiable maps $f : U_1 \rightarrow V_1$ and $g : U_1 \rightarrow GL(\mathbf{R}^k)$. A **vector bundle atlas** is a cover of compatible vector bundle parametrizations. A **vector bundle** is a set E equipped with a vector bundle atlas.

Properties of Vector Bundles.

- (1) Define $G \subset E$ to be **open** if $X^{-1}(G \cap X(U))$ is open in \mathbf{R}^{n+k} for all vector bundle parametrizations (X, U) . This is the unique topology on E making the maps X homeomorphisms.
- (2) A vector bundle is an $(n+k)$ -dimensional manifold.
- (3) Given two parametrizations (X, U) and (Y, V) with $\xi \in X(U) \cap Y(V) \subset E$, suppose $\xi = X(a, b)$ and $\xi = Y(c, d)$. Then $(c, d) = (f(a), g(a)b)$ and so for $t \in \mathbf{R}$, we have $X(a, tb) = Y(c, td)$, since $Y^{-1}(X(a, tb)) = Y^{-1} \circ X(a, tb) = (f(a), g(a)(tb)) = (f(a), tg(a)b) = (c, td)$. Thus we may define an **action** of \mathbf{R} on E by setting $t\xi = X^{-1}(a, tb)$. This defines a map $\mathbf{R} \times E \rightarrow E$ satisfying $t(s\xi) = (ts)\xi$ for $\xi \in E$ and $t, s \in \mathbf{R}$. This is **scalar multiplication**. Note that for any parametrization (X, U) , $tX(U) \subset X(U)$.
- (4) Define $M = 0.E = \{0\xi : \xi \in E\}$ and let $\pi_1 : \mathbf{R}^{n+k} \rightarrow \mathbf{R}^n$ be the projection onto \mathbf{R}^n . Then M is a subset of E and for any vector bundle parametrization $(X, U = U_1 \times \mathbf{R}^k)$ on E , the map $x : U_1 \rightarrow M$ defined by $x(a) = X(a, 0)$ defines a manifold parametrization (x, U_1) of M , and $x^{-1}(\xi) = \pi_1(X^{-1}(\xi))$. Any two of these are compatible manifold parametrizations into M , since $y^{-1} \circ x(a) = \pi_1(Y^{-1}(X(a, 0))) = \pi_1(f(a), 0) = f(a)$. Thus M is an n -dimensional manifold.
- (5) Define $\pi : E \rightarrow M$ by setting $\pi(\xi) = 0\xi$. Then π is a smooth map since for a vector bundle parametrization (X, U) on E and corresponding manifold parametrization (x, U_1) on M , we have

$$x^{-1} \circ \pi \circ X(a, b) = \pi_1(X^{-1}(X(a, b))) = a.$$

Moreover this formula shows that π is onto and a **submersion** (π_* is onto). Since x and X are homeomorphisms, this also shows $\pi : E \rightarrow M$ is an open map.

- (6) For each parametrization (X, U) , $\pi^{-1}(\pi(X(U))) = X(U)$. Indeed, let $\xi \in E$ with $\pi(\xi) = 0, \xi \in U$. Then, by continuity, $t\xi \in U$ for t small and positive. Hence $\xi = (1/t)t\xi \in U$.
- (7) For each $p \in M$, let $E_p = \pi^{-1}(p)$. Let (X, U) be a parametrization at $p \in M \subset E$, so $p \in \pi(X(U))$. From the previous step, we have $E_p \subset U$. Thus given $\xi \in E_p$ and $\eta \in E_p$ with $\xi = X(a, b)$, $\eta = X(a, c)$, we can define $\xi + \eta = X(a, b + c)$, which can be checked to be independent of the parametrization as in scalar multiplication above. This is **addition** on the **fibers**. It follows that E_p is a k -dimensional vector space, for each $p \in M$.

Thus we conclude a vector bundle is a manifold consisting of a disjoint union of vector spaces; another way of thinking of a vector bundle is as a family of vector spaces parametrized by the set M . In this context M is called the **base** space and E is the **total** space. Often a vector bundle is defined to be a map $\pi : E \rightarrow M$ between two manifolds such that there is a manifold atlas $\{(x, U_1)\}$ on M and a vector-bundle atlas $\{(X, U)\}$ on E such that $\pi^{-1}(x(U_1)) = X(U)$ for every parametrization (x, U_1) . As we have seen above, M and π exist as soon as we have a vector-bundle atlas, there is no need to specify them separately.