# Orientations on Manifolds

Let $V$ be an $n$-dimensional real vector space. Let $\mathbf{B}(V)$ denote the set of bases on $V$. If $e = (e_1, \ldots, e_n)$ and $f = (f_1, \ldots, f_n)$ are bases of $V$, there is an invertible matrix $a = (a_{ij})$ such that

$$f_i = \sum_{j=1}^{n} a_{ij} e_j.$$

We say $e$ and $f$ *have the same orientation* if $\det(a) > 0$.

**Exercise 1.** *Having the same orientation is an equivalence relation on $\mathbf{B}(V)$ and there are exactly two equivalence classes.*

The equivalence classes are called *orientations on $V$*. An *oriented vector space* is a pair $(V, \mathcal{O})$ with $V$ a vector space and $\mathcal{O}$ an orientation on $V$. If $\mathcal{O}$ is an orientation on $V$, we define $-\mathcal{O}$ to be the other orientation (remember, there are exactly two orientations). If $e$ is a basis of $V$, we let $[e]$ denote the orientation containing $e$.

Let $M$ be a manifold. Then to each $p \in M$ is associated the tangent space $T_pM$, which is a finite-dimensional vector space. An *orientation on $M$* is a choice of orientation $\mathcal{O}_p$ on $T_pM$ for each $p \in M$ that is differentiable. By differentiable, we mean for each parametrization $(x, U)$ of $M$ *such that $U$ is connected*, either

$$\mathcal{O}_p = \left[ \left.\frac{\partial}{\partial x_1}\right|_p, \ldots, \left.\frac{\partial}{\partial x_n}\right|_p \right], \quad \forall p \in x(U),$$

or

$$\mathcal{O}_p = -\left[ \left.\frac{\partial}{\partial x_1}\right|_p, \ldots, \left.\frac{\partial}{\partial x_n}\right|_p \right], \quad \forall p \in x(U)$$

(remember, $[e]$ denotes the class containing the basis $e$). We call the parametrization $(x, U)$ *positive* or *negative* depending on whether the first or second case holds above.

**Exercise 2.** *If $M$ is connected and $\mathcal{O}_1$ and $\mathcal{O}_2$ are orientations on $M$, then either $\mathcal{O}_1 = \mathcal{O}_2$ at all points or $\mathcal{O}_1 = -\mathcal{O}_2$ at all points.*

The proof is by showing the set $S = \{p \in M : \mathcal{O}_{1p} = \mathcal{O}_{2p}\}$ is both open and closed, hence either empty or equal to $M$.

An *oriented manifold* is a pair $(M, \mathcal{O})$ where $M$ is a manifold and $\mathcal{O}$ is an orientation on $M$. We showed above that a connected manifold has at most two orientations. If $M$ has at least one orientation, we say $M$ is *orientable.* If $M$ is orientable and connected, we get exactly two orientations on $M$.

Now recall the definition of *differentiable structure* in the book (page 2). We say a differentiable structure is *oriented* if, for every pair of parametrizations $(x, U)$ and $(y, V)$ in the differentiable structure, the differential of $y^{-1} \circ x$ at $p$ has positive determinant for every point $p$ in the intersection of the coordinate charts $x(U) \cap y(V)$. The book (page 18) defines an orientation to be an oriented differentiable structure.

**Exercise 3.** *Show that the definition of orientation given above and the book's definition are equivalent.*

**Exercise 4.** *Show that a Lie group is orientable.*

THEOREM **1.** *Let $M \subset \mathbf{R}^n$ be a hypersurface. Then $M$ is orientable iff there is a unit normal on $M$, i.e. a differentiable map $N : M \to \mathbf{R}^n$ such that $N(p) \perp T_pM$ for every $p \in M$ and $|N(p)| = 1$.*

*Proof.* Given such a normal $N$, we say a basis $e = (e_1, \ldots, e_{n-1})$ of $T_pM$ is positive if the basis $(e_1, \ldots, e_{n-1}, N(p))$ is a positive basis in $\mathbf{R}^n$. Since, for any parametrization $(x, U)$ of $M$,

$$\left\{ \frac{\partial}{\partial x_1}\bigg|_p, \ldots, \frac{\partial}{\partial x_n}\bigg|_p, N(p) \right\}$$

is either a positive basis in $\mathbf{R}^n$ for all $p \in x(U)$, or a negative basis in $\mathbf{R}^n$ for all $p \in x(U)$, this defines an orientation on $M$.

Conversely, assume $M$ is orientable and let $p_0 \in M$; since $M$ is a hypersurface, there is a neighborhood $V$ of $p_0$ in $\mathbf{R}^n$, an open set $U$ in $\mathbf{R}^{n-1}$, a $\delta > 0$, and a diffeomorphism $\phi : V \to U \times (-\delta, \delta) \subset \mathbf{R}^n$ such that $\phi(V \cap M) = \phi(V) \cap \{t : t_n = 0\}$. Let $f$ be the $n$-th component of $\phi$. Then $f : V \to \mathbf{R}$ is differentiable, $f^{-1}(0) = M \cap V$, and $df \neq 0$ on $M \cap V$. Moreover, $x(t_1, \ldots, t_{n-1}) = \phi^{-1}(t_1, \ldots, t_{n-1}, 0)$ yields a parametrization $x : U \to M$. Let $N(p) = df(p)/|df(p)|$ if $(x, U)$ is a positive parametrization, and let $N(p) = -df(p)/|df(p)|$ if $(x, U)$ is a negative parametrization. Then $N : M \cap V \to \mathbf{R}^n$ is a differentiable unit normal and $\{e_1, \ldots, e_{n-1}, N(p)\}$ is a positive basis in $\mathbf{R}^n$ for any positive basis $e$ in $T_pM$. If $N'$ is another differentiable unit normal obtained from an overlapping neighborhood $V'$, then $N = N'$ on the intersection since $\{e_1, \ldots, e_{n-1}, N(p)\}$ and $\{e_1, \ldots, e_{n-1}, N'(p)\}$ are both positive. □

THEOREM **2.** *Let $M \subset \mathbf{R}^n$ be a compact connected hypersurface. Then $M$ is orientable iff there is a differentiable function $f : \mathbf{R}^n \to \mathbf{R}$ such that $M = f^{-1}(0)$ and $|df| = 1$ on $M$.*

Thus orientable hypersurfaces are exactly the level sets corresponding to regular values of differentiable functions, at least when they're compact and connected.

*Proof.* Of course if we are given $f$, we take $N = df$, it is the converse that is the issue. Let $N : M \to \mathbf{R}^n$ be a positive unit normal; we seek a function $f : \mathbf{R}^n \to \mathbf{R}$ such that $f^{-1}(0) = M$ and $df = N$ on $M$. Define $\alpha : \mathbf{R} \times M \to \mathbf{R}^n$ by $\alpha(t, p) = p + tN(p)$. Then for each $p \in M$, the differential of $\alpha$ at $(p, 0)$ is an isomorphism hence by the inverse function theorem, $\alpha$ is a diffeomorphism on $(-\delta, \delta) \times U$ for some neighborhood $U$ of $p$ in $M$ and some $\delta > 0$. Since $M$ is compact, finitely many such neighborhoods cover $M$. Let $\delta_0$ be the minimum of the corresponding $\delta$'s. We claim there is a $\delta < \delta_0$ such that $\alpha : (-\delta, \delta) \times M \to \mathbf{R}^n$ is injective. If not, there is a sequence $\delta_n \to 0$ and $s_n$ and $t_n$ with $|s_n| < \delta_n$ and $|t_n| < \delta_n$ and $p_n$ and $q_n$ in $M$ satisfying $(t_n, p_n) \neq (s_n, q_n)$ and

$$\alpha(t_n, p_n) = \alpha(s_n, q_n)$$

for all $n$. Note the formula for $\alpha$ implies we have $p_n \neq q_n$. By compactness of $M$, passing to a subsequence, we get $p_n \to p$, $q_n \to q$ and $s_n \to 0$ and $t_n \to 0$. Hence $p = q$ so for $n$ sufficiently large both $p_n$ and $q_n$ lie in the neighborhood $U$ containing $p = q$, contradicting the injectivity of $\alpha$ on $(-\delta, \delta) \times U$.

Thus we have constructed a $\delta > 0$ and a neighborhood $V$ of $M$ in $\mathbf{R}^n$ such that $\alpha : (-\delta, \delta) \times M \to V$ is a diffeomorphism. Since $M$ is connected, so is $(-\delta, \delta) \times M$ hence so is $V$. For each $x \in V$, let $\tau(x)$ and $\pi(x)$ be the unique $-\delta < t < \delta$ and

$p \in M$ satisfying $\alpha(t,p) = x$. Then $\tau : V \to \mathbf{R}$ is differentiable and $\tau^{-1}(0) = M$, hence $d\tau(p) \perp T_pM$ for all $p \in M$. Differentiating $\tau(p + tN(p)) = t$ at $t = 0$ yields $d\tau(p) \cdot N(p) = 1$, hence $d\tau(p) = N(p)$ for all $p \in M$. In fact $\tau$ satisfies all the requirement for $f$ except that it is defined only on a neighborhood of $M$, not on all of $\mathbf{R}^n$. To extend $\tau$ to $\mathbf{R}^n$, we will need Poincaré's Lemma which we derive below.

Let $V_s = \{x \in V : |\tau(x)| < s\}$; then $V = V_\delta$. Construct a function $h : \mathbf{R} \to \mathbf{R}$ that satisfies

(1) $h$ is differentiable,
(2) $h$ is odd and increasing,
(3) $h(t) = t$ for $0 < t < \delta/2$,
(4) $h(t) = 3\delta/4$ for $t > 3\delta/4$.

(draw a picture of $h$.) Define $g : V_\delta \to \mathbf{R}$ by $g = h \circ \tau$. Then $g = \tau$ on $V_{\delta/2}$ and $g = 3\delta/4$ on $V^c_{3\delta/4}$.

Let $X = (X_1, \ldots, X_n)$ be a vector field on an open set $V$ in $\mathbf{R}^n$. We say $X$ is *exact* if $X = dg$ for some differentiable $g : V \to \mathbf{R}$, and we say $X$ is *closed* if

$$\frac{\partial X_j}{\partial x_i} = \frac{\partial X_i}{\partial x_j}, \qquad i \neq j,$$

on $V$. Since

$$\frac{\partial X_j}{\partial x_i} = \frac{\partial^2 g}{\partial x_i \partial x_j} = \frac{\partial^2 g}{\partial x_j \partial x_i} = \frac{\partial X_i}{\partial x_j},$$

exact implies closed. Poincaré's Lemma states that closed implies exact, at least when $V = \mathbf{R}^n$ (this is false for some open sets $V$, for example $V = \mathbf{R}^2 \setminus 0$ and $X = dz/z$ ).

Continuing with the proof, let $X = dg$ on $V_\delta$; then $X$ is exact on $V_\delta$ hence $X$ is closed on $V_\delta$. Since $X$ is compactly supported in $V_\delta$, $X$ extends to a closed vector field on all of $\mathbf{R}^n$. By Poincaré's Lemma, there is a differentiable $f : \mathbf{R}^n \to \mathbf{R}$ such that $X = df$ on $\mathbf{R}^n$. By adding a constant to $f$, we may assume $f = 0$ at some point in $M$. We claim this $f$ is what we seek: Indeed, $X = dg = df$ on $V_\delta$; since $V_\delta$ is connected, $f - g$ is a constant on $V_\delta$, hence $f = g$ on $V_\delta$. Moreover $df = 0$ on $V^c_{3\delta/4}$; we use this to show $f = \pm 3\delta/4$ on $V^c_{3\delta/4}$. Given $x \notin V_{3\delta/4}$, choose a point $p \in M$ that is closest to $x$. Then the line segment $[p, x]$ is orthogonal to $T_pM$. Let $q$ be the first point on $[p, x]$ such that $[q, x] \subset V^c_{3\delta/4}$. Then $[p, x] \cap V_{3\delta/4} = [p, q)$ and $df = 0$ on $[q, x]$. Thus $f$ is constant on $[q, x]$, $f(q) = \pm 3\delta/4$, hence $f(x) = \pm 3\delta/4$. □

An open set $V \subset \mathbf{R}^n$ is *star-shaped* if $x \in V$ implies the line segment $[0, x] \subset V$. Given $f : \mathbf{R}^n \to \mathbf{R}$, note the derivative of $f$ along the line segment $[0, x]$ is

$$\frac{d}{dt} f(tx) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(tx) x_i.$$

Of course $\mathbf{R}^n$ is star-shaped.

Theorem **3.** *(Poincaré's Lemma) If $X$ is closed on a star-shaped open set $V$, then $X$ is exact on $V$.*

*Proof.* Let $f(x)$ denote the line integral of $X$ over $[0, x]$,

$$f(x) = \int_{[0,x]} X = \int_0^1 \langle X(tx), x \rangle \, dt = \int_0^1 \left( \sum_{i=1}^{n} X_i(tx) x_i \right) dt.$$

Then

$$\begin{aligned}\frac{\partial f}{\partial x_j}(x) &= \int_0^1 X_j(tx)\,dt + \int_0^1 \left(\sum_{i=1}^n \frac{\partial X_i}{\partial x_j}(tx)tx_i\right)\,dt \\ &= \int_0^1 X_j(tx)\,dt + \int_0^1 \left(\sum_{i=1}^n \frac{\partial X_j}{\partial x_i}(tx)tx_i\right)\,dt \qquad (1)\\ &= \int_0^1 X_j(tx)\,dt + \int_0^1 t\frac{dX_j}{dt}(tx)\,dt = \int_0^1 \frac{d}{dt}\left(tX_j(tx)\right)\,dt = X_j(x).\end{aligned}$$

□

As a corollary, we see that *the complement of a connected compact orientable hypersurface in* $\mathbf{R}^n$ *has two components.* One of these components, the *inside*, is bounded, while the other, the *outside*, is unbounded.

The Jordan curve theorem says that the complement of a simple closed *continuous* curve in $\mathbf{R}^2$ has two components, but this is a deeper theorem.